Pi Math Contest Fermat Division

Final Round - 2026 Solutions

Solutions

1. Evaluate $48 \div 6 + 6 \times 6 - 6$.

   Answer (38): Following the order of operations,

   $$(48 \div 6) + (6 \times 6) - 6 = 8 + 36 - 6 = 38.$$

2. The average of nine consecutive even integers is 50. What is the largest of these integers?

   Answer (58): The middle (fifth) integer equals the average, 50. The nine integers are $42, 44, 46, 48, 50, 52, 54, 56, 58$. The largest is 58.

3. What is the value of $\frac{9^4 \times 2^4}{6^4}$?

   Answer (81): Since $9^4 = 3^8$ and $6^4 = 2^4 \cdot 3^4$, we have

   $$\frac{3^8 \cdot 2^4}{2^4 \cdot 3^4} = 3^4 = 81.$$

4. How many subsets of $\{1,2,3,4,5,6\}$ contain at least one even number?

   Answer (56): We recall that given any (finite) set A, the number of subsets of A is $2^{|A|}$. This implies that there are $2^6 = 64$ subsets of $\{1,2,3,4,5,6\}$. We subtract the number of subsets that do not contain any even number; these are precisely the $2^3 = 8$ subsets of $\{1,3,5\}$. We conclude that there are $64 - 8 = 56$ subsets which contain at least one even number.

5. A rectangular box with dimensions 2, 5, and 7 has no lid. What is the least possible exterior surface area of the box?

   Answer (83): The surface area of the box with the lid is $2(2 \cdot 5) + 2(2 \cdot 7) + 2(5 \cdot 7) = 118$. Now, to make the surface area of the box as small as possible, the area of the lid should be as large as possible, so the lid should be a 5×7 face. So, the area of the lid is $5 \cdot 7 = 35$ and the surface area of the box is $118 - 35 = 83$.

6. What is the value of

   $$95 \times 96 + 97 \times 98 - 2 \times 95 \times 98?$$

   Answer (6): Let $a = 95$. Then the expression is equivalent to evaluating $a(a+1) + (a+2)(a+3) - 2a(a+3)$, which simplifies to

   $$(a^2 + a) + (a^2 + 5a + 6) - (2a^2 + 6a) = 6.$$

   Note that the value of this expression is independent of a.

7. The four-digit number $\overline{1A1B}$ is divisible by 36. What is the largest possible value of $A + B$?

   Answer (7): Divisibility by 36 requires divisibility by both 4 and 9.

   For divisibility by 4, the last two digits $\overline{1B}$ must form a multiple of 4, so $B \in \{2,6\}$.

   For divisibility by 9, the digit sum $1 + A + 1 + B = 2 + A + B$ must be a multiple of 9, so $A + B \in \{7,16\}$.

   The value $A + B = 16$ is not achievable since $B \leq 6$ and $A \leq 9$.

   The value $A + B = 7$ is achievable, for instance with $B = 6$ and $A = 1$.

   Therefore, the answer is 7.

8. Barbara places two "B" tiles, two "R" tiles, and three "A" tiles into a bag. She selects two of these tiles at random, without replacement. To the nearest whole percent, what is the probability that both of these tiles are "A" tiles?

   Answer (14): There are $2 + 2 + 3 = 7$ tiles, so there are $\binom{7}{2} = 21$ ways that Barbara can select two of these tiles. Of these, there are 3 ways where both tiles are "A" tiles ($\{A_1, A_2\}$, $\{A_2, A_3\}$, $\{A_1, A_3\}$, where A_1, A_2, and A_3 represent the three "A" tiles). The probability is $\frac{3}{21} = \frac{1}{7}$, which is approximately 14 percent.

9. Ronaldo jogged 10% faster on Tuesday than he did on Monday, and jogged for 6 fewer minutes on Tuesday than he did on Monday. Given that he jogged the same distance on both days, how many minutes did Ronaldo jog on Monday?

   Answer (66): Suppose that Ronaldo jogged for t minutes on Monday at an average speed of r (the units of r are irrelevant). Then on Tuesday, he jogged for $t - 6$ minutes at an average speed of $1.1r$. Since he covered same distances both days, using $d = rt$, we have

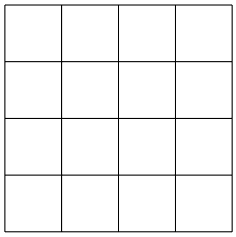
   $$rt = 1.1r(t - 6).$$

   Dividing both sides by r gives $t = 1.1(t - 6)$. Solving this equation for t gives $t = 66$, so Ronaldo jogged for 66 minutes on Monday.

10. What is the largest 2-digit factor of the number 111,111?

    Answer (91): We write $111111 = 111 \times 1001$. The prime factorizations of 111 and 1001 are 3×37 and $7 \times 11 \times 13$, respectively, so the prime factorization of 111111 is $3 \times 7 \times 11 \times 13 \times 37$. The largest 2-digit factor of this number is $7 \times 13 = 91$.

11. The figure below shows a 4×4 grid of unit squares. How many rectangles formed by the grid lines have an even area?



Answer (64): A rectangle in a 4×4 grid is determined by choosing two of the 5 vertical grid lines and two of the 5 horizontal grid lines.

Thus, the total number of rectangles is

$$\binom{5}{2}\binom{5}{2} = 10 \cdot 10 = 100.$$

A rectangle has odd area if and only if both its side lengths are odd.

The possible side lengths are $1, 2, 3$, and 4. There are

$$4 + 2 = 6$$

horizontal segments of odd length:

$$4 \text{ of length } 1, \quad 2 \text{ of length } 3.$$

Similarly, there are 6 vertical segments of odd length.

Therefore, the number of rectangles with odd area is

$$6 \cdot 6 = 36.$$

Hence, the number of rectangles with even area is

$$100 - 36 = 64.$$

12. Compute the following expression:

$$\sqrt[3]{1320 + \sqrt[3]{1320 + \sqrt[3]{1320 + \dots}}}$$

Answer (11): Let the expression be S. Then $S = \sqrt[3]{1320 + S}$, so $S^3 = 1320 + S$ or

$$(S-1)S(S+1) = 1320 = 10 \cdot 11 \cdot 12,$$

which yields the answer $S = 11$.

For uniqueness in real numbers, note that

- for $S < -1$, we have $f(S) < 0$
- for $-1 \leq S \leq 1$, we have $f(S) \leq 4$ and
- for $S > 1$, the function $f(S) = S(S^2 - 1)$ is increasing so, there is only one solution here for $f(S) = 1320$.

13. Five children sit in five seats around a circular table. Two of them, Amelie and Emilia, are twins and will sit next to each other. In how many ways can the children be seated? (Seats are distinct, so rotations count as different arrangements.)

Answer (60): Fix one twin (Amelie) at the top of the circle. The other twin (Emilia) has 2 possibilities. There are 6 ways to rearrange the rest of the children. Accounting for 5 rotations, the answer is $2 \cdot 6 \cdot 5 = 60$.

14. Regular polygons P and Q are such that P has 32 more sides than Q. Each interior angle of P measures 15° more than each interior angle of Q. How many sides does P have?

Answer (48): Let n and $n - 32$ be the number of sides of P and Q, respectively. Since, the interior angles differ by 15°, the exterior angles differ by 15° as well and we have

$$\frac{360}{n-32} - \frac{360}{n} = 15.$$

This leads to

$$(n-32)n = 24 \times 32 = 16 \times 48$$

and $n = 48$.

15. A paper towel manufacturer produces rolls which have an inner and outer diameter of 1 inch and 3 inches as shown in Figure 1. A second manufacturer produces "mega" rolls, shown in Figure 2, and has three times the

number of paper towels per roll than the roll in Figure 1. Assume that the number of paper towels per roll is directly proportional to its cross-sectional area. Rounded to the nearest whole percent, the outer diameter of the mega roll is how many percent larger than the outer diameter of the roll in Figure 1?

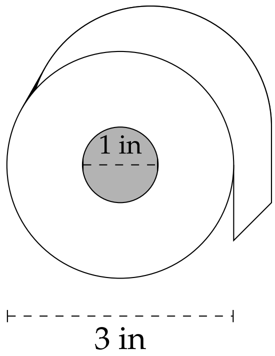


Figure 1

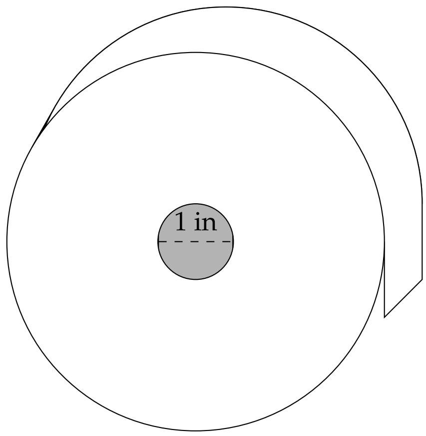


Figure 2

Answer (67): The cross-sectional area of the roll in Figure A is $\frac{1}{4}(3^2\pi - 1^2\pi) = 2\pi$ square inches. Therefore, the cross-sectional area of the roll in Figure B is $2\pi \times 3 = 6\pi$ square inches. Letting d be the outer diameter of the mega roll, we have $\frac{1}{4}(d^2\pi - 1^2\pi) = 6\pi$; solving for d gives $d = 5$. Therefore, the mega roll has an outer diameter of 5 inches, which is $66.\overline{6}\% \approx 67\%$ larger than the outer diameter (3 inches) of the roll in Figure A.

16. What is the smallest positive integer whose number of composite divisors is itself a composite number?

    Answer (30): Observe that 30 works because it has 4 composite divisors: 6, 10, 15, and 30. Next we show that 30 is the smallest such number.

    The number n must have at least 4 composite divisors. If it has exactly k distinct prime factors, it must have at least $k + 5$ divisors. Let's do casework based on k:

    - $k = 1$: $n = p^a$. Since $a + 1 \geq k + 5 = 6$, we have $a \geq 5$ and $n \geq 2^5 > 30$.

- $k = 2$: $n = p^a q^b$ with $a \geq b$. We have $(a+1)(b+1) \geq k+5 = 7$. If $b = 1$ then $a \geq 3$. Only $n = 2^3 \cdot 3 = 24$ is less than 30. But 24 has 5 composite divisors and 5 is not composite. If $b \geq 2$, then since $a \geq 2$, we have $n \geq 2^2 \cdot 3^2 > 30$.
- $k \geq 3$. Then $n \geq 2 \cdot 3 \cdot 5 \geq 30$.

We conclude that the answer is 30.

17. Let N be the number of ordered triples (a, b, c) of non-negative integers such that
$$a \cdot b + b \cdot c = 360.$$
What are the rightmost two digits of N?

Answer (94): Rewrite the given equation as $b(a+c) = 360$. This implies that b must be a positive factor of 360, and that $a+c$ must also be a positive factor of 360 which is equal to $\frac{360}{b}$.

In general, given a number k, the number of non-negative solutions to the equation $a + c = k$ is $k+1$ (since (a, c) can be any of $(0, k)$, $(1, k-1)$, ..., $(k, 0)$). Therefore, for any factor b of 360, there are $\frac{360}{b} + 1$ ways to choose a and c such that $a + c = \frac{360}{b}$. The number of non-negative solutions is equal to the sum over all factors b of 360 of the expression $\frac{360}{b} + 1$; mathematically, this is written as follows:

$$\begin{aligned} N &= \sum_{b|360} \frac{360}{b} + 1 \\ &= \sum_{b|360} \frac{360}{b} + \sum_{b|360} 1 \\ &= \sigma(360) + d(360) \end{aligned}$$

where $\sigma(n)$ and $d(n)$ denote the sum of the positive divisors of n and the number of divisors of n, respectively. The prime factorization of 360 is $2^3 \times 3^2 \times 5^1$, so

$$\sigma(360) = (1+2+4+8)(1+3+9)(1+5) = 1170$$

and
$$d(360) = (3+1)(2+1)(1+1) = 24.$$

Then $N = 1170 + 24 = 1194$, and the rightmost two digits are 94.

18. When $\sqrt{10001}$ is written as a decimal, what are the first two nonzero digits to the right of the decimal point? For example, the first two nonzero digits to the right of the decimal point in the number 3.01065 are 16.

    Answer (49): Noting that $\sqrt{10000} = 100$, let $\sqrt{10001} = 100 + r$ for some $r > 0$. Note that r is relatively small since $10001 \approx 10000$; specifically, it is not difficult to show that $r < 0.01$. Squaring both sides gives $10001 = 10000 + 200r + r^2$, so $1 = 200r + r^2$. Since r is small, $r^2 \approx 0$, and $200r + r^2 \approx 200r$. Then $r \approx \frac{1}{200} = 0.005$; since $200r$ is marginally less than 1, we can reasonably conclude that $r \approx 0.0049$, and that the first two nonzero digits are 49.

    Remark: $\sqrt{10001} \approx 100.004999875$.

19. Lothar writes the number 1023 on a blackboard. On each step, he does the following: if the number n on the blackboard is odd, he erases it and writes the number $3n + 1$. Otherwise, if n is even, he erases it and writes the number $\frac{n}{2}$. For example, after 2 steps, Lothar will have written the number 3070, then the number 1535. What are the last two digits of the number that will be on the blackboard after 20 steps, starting with the number 1023?

    Answer (48): Let (a_n) denote the resulting sequence with $a_0 = 1023$; we wish to find the last two digits of a_{20}. The key observation is to recognize that $1023 = 2^{10} - 1$, and that the first several terms in this sequence, starting with $2^{10} - 1$, obey a pattern.

$$
\begin{aligned}
a_0 &= 2^{10} - 1 \\
a_1 &= 3(2^{10} - 1) + 1 = 3 \cdot 2^{10} - 2 \\
a_2 &= 3 \cdot 2^9 - 1 \\
a_3 &= 3(3 \cdot 2^9 - 1) + 1 = 3^2 \cdot 2^9 - 2 \\
a_4 &= 3^2 \cdot 2^8 - 1 \\
a_5 &= 3(3^2 \cdot 2^8 - 1) + 1 = 3^3 \cdot 2^8 - 2 \\
a_6 &= 3^3 \cdot 2^7 - 1 \\
&\vdots
\end{aligned}
$$

In general, $a_{2k} = 3^k 2^{10-k} - 1$ where $0 \le k \le 10$. Then a_{20} will equal $3^{10} - 1$. We can either recall $3^{10} = 59049$, or use the fact that $3^{10} = (3^5)^2 \equiv 43^2 \equiv 49 \pmod{100}$, so the last 2 digits of $a_{20} = 3^{10} - 1$ are 48.

Remark: The *Collatz conjecture*, named after Lothar Collatz, hypothesizes that given any starting positive integer, the resulting sequence will eventually reach the number 1. As of 2026, this conjecture remains unproven.

20. Call a positive integer a *phoenix* if its last two digits are unchanged when the integer is cubed. What is the largest phoenix less than 99?

Answer (76): Let the answer be n. Then we have

$$n^3 - n \equiv n(n-1)(n+1) \equiv 0 \pmod{4},$$
$$n^3 - n \equiv n(n-1)(n+1) \equiv 0 \pmod{25}.$$

From these relations we derive

$$n \equiv 0, 1, 3 \pmod{4},$$
$$n \equiv 0, 1, 24 \pmod{25}.$$

Since n must be within 1 of a multiple of 25, the largest candidate for $n < 99$ is 76, which satisfies both mod 4 and mod 25 conditions. Hence, the answer is 76.

21. Chris the chameleon starts at $(0,0)$ and wants to reach his house at $(6,4)$. Each step moves him either 1 unit right or 1 unit up. The points $(2,2)$ and $(5,3)$ are burned and cannot be visited. In how many such paths can Chris reach his house?

Answer (56): Let $A = (0,0)$, $B = (6,4)$, $P = (2,2)$ and $Q = (5,3)$.

The number of all shortest paths from A to B is $\binom{10}{4} = 210$.

Among these paths:

$\binom{4}{2} \cdot \binom{6}{4} = 90$ of them pass through P ($A \to P \to B$),
$\binom{8}{3} \cdot \binom{2}{1} = 112$ of them pass through Q ($A \to Q \to B$), and
$\binom{4}{2} \cdot \binom{4}{1} \cdot \binom{2}{1} = 48$ of them pass through both P and Q ($A \to P \to Q \to B$).

By the Principle of Inclusion and Exclusion, the number of shortest paths passing through neither P nor Q is

$$210 - 90 - 112 + 48 = 56.$$

22. Harry and Ted are playing a game. Harry constantly flips a fair coin until he gets two heads in a row, in which case he wins, or he gets three tails in a row, in which case Ted wins. The probability that Harry wins is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

    Answer (17): Let a be the probability that Harry wins given a streak of one head, and let b be the probability that Harry wins given a streak of one tails. In the case of the streak with one head, we have $a = \frac{1}{2} + \frac{1}{2}b$ because Harry either wins with the second head with probability $1/2$ or gets to one tail with probability $1/2$. In the case of the streak with one tails, we have $b = \frac{a}{2} + \frac{a}{4} = \frac{3}{4}a$ because Harry's only chance of winning comes from flipping a heads within the next two flips, which occurs with probability $3/4$. Solving gives $a = \frac{4}{5}$ and $b = \frac{3}{5}$. Thus, the probability that Harry wins is $\frac{1}{2} \cdot \frac{4}{5} + \frac{1}{2} \cdot \frac{3}{5} = \frac{7}{10}$, so $m + n = 7 + 10 = 17$.

23. What are the first (leftmost) two digits of 99^{20} when written in decimal notation?

    Answer (81):

    $$99^{20} = (100 - 1)^{20}.$$

    Using the Binomial Theorem,

    $$(100-1)^{20} = 100^{20} - 20 \cdot 100^{19} + \binom{20}{2} 100^{18} - \cdots .$$

    Factoring out 100^{18}, we get

    $$99^{20} = 100^{18} \left(100^2 - 20 \cdot 100 + \binom{20}{2} - \cdots \right).$$

    The first three terms inside the parentheses are

$$100^2 - 20 \cdot 100 + \binom{20}{2} = 10000 - 2000 + 190 = 8190.$$

The remaining terms begin with

$$-\binom{20}{3} 100^{-1} = -\frac{1140}{100} = -11.4,$$

and all later terms have even smaller magnitude. Therefore, the quantity inside the parentheses is between 8100 and 8200.

Thus,

$$99^{20} = 100^{18} \times (\text{a number between 8100 and 8200}),$$

so the first two digits of 99^{20} are 81.

24. Real numbers x and y satisfy

$$2x^2 + 5y^2 + 2xy + 13 = 2x + 16y.$$

Find the value of $2x^2 + y^2$.

Answer (3): The problem is impossible if we try to just take the one equation and manipulate it to get $2x^2 + y^2$. Instead, there must be some hidden subtlety here that fixes the value of $2x^2 + y^2$. If we plug some values in to the equation, we may notice that the LHS is always larger than the RHS. Indeed, if we consider the equation as a quadratic in x and complete the square, we can rearrange the equation to

$$(2x - 1 + y)^2 + (3y - 5)^2 = 0.$$

Now, it is clear that $(x, y) = (-\frac{1}{3}, \frac{5}{3})$ so the answer is 3.

Alternate Ending: As above, we are motivated to solve for the values of x and y. We can use the quadratic formula on x: we have

$$2x^2 + (2y - 2)x + (y^2 - 11y + 13) = 0,$$

so

$$x = \frac{1 - y \pm \sqrt{-(3y-5)^2}}{2}.$$

Since the discriminant is nonpositive it must be equal to 0, so $y = \frac{5}{3}$. Then, we can solve for $x = -\frac{1}{3}$ and the answer is 3.

25. A rectangular box has dimensions that are distinct positive integers. Its volume is numerically equal to twice its surface area. The sum of its three dimensions is 56. Find the length of the box's space diagonal, rounded to the nearest whole number.

Answer (37): Let the dimensions be a, b, c. Volume equals twice surface area:

$$abc = 2 \cdot 2(ab + ac + bc) \ = \ 4ab + 4ac + 4bc.$$

A useful factorization. Expanding,

$$(a-4)(b-4)(c-4) = abc - 4(ab + ac + bc) + 16(a + b + c) - 64.$$

Since $abc = 4(ab + ac + bc)$, the first two terms cancel, leaving

$$(a-4)(b-4)(c-4) = 16(a + b + c) - 64 = 16(a + b + c - 4).$$

Apply the sum condition. With $a + b + c = 56$,

$$(a-4)(b-4)(c-4) = 16(56-4) = 16 \cdot 52 = 64 \cdot 13.$$

Substitute. Let $m = a - 4,\ n = b - 4,\ r = c - 4$. Then

$$mnr = 64 \cdot 13, \qquad m + n + r = 56 - 12 = 44.$$

Since 13 is prime and divides mnr, one factor is a multiple of 13. As each factor is at most 44, that factor (say r) is 13, 26, or 39.

If $r = 13$, then $mn = 64$ and $m + n = 31$ have no integer solutions.

If $r = 26$, then $mn = 32$ and $m + n = 18$. This leads to $\{m, n\} = \{16, 2\}$.

If $r = 39$, then mn is not an integer.

Recover the dimensions. From $r = 26$ we get (without loss of generality) $m = 2,\ n = 16,\ r = 26$, so

$$(a, b, c) = (m + 4, n + 4, r + 4) = (6, 20, 30).$$

Space diagonal.

$$\sqrt{6^2 + 20^2 + 30^2} = \sqrt{36 + 400 + 900} = \sqrt{1336} \approx \boxed{37}.$$