# Pi Math Contest Gauss Division

2025 Solutions

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## Solutions

1. What is the value of $\sqrt{21+\sqrt{13+\sqrt{7+\sqrt[3]{8}}}}$?

   **Answer (5):** Simplify the innermost expressions first:

$$\begin{aligned}\sqrt{21+\sqrt{13+\sqrt{7+\sqrt[3]{8}}}} &= \sqrt{21+\sqrt{13+\sqrt{7+2}}} \\ &= \sqrt{21+\sqrt{13+3}} \\ &= \sqrt{21+4} \\ &= 5.\end{aligned}$$

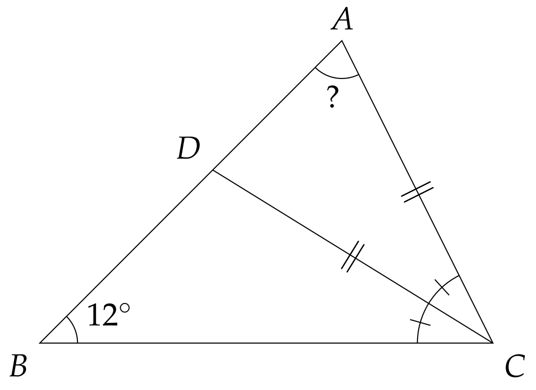
2. How many digits does the number $32^{11} \cdot 125^{15}$ have in its decimal representation?

   **Answer (49):** Rewrite the given number using exponents whose bases are 2 and 5:

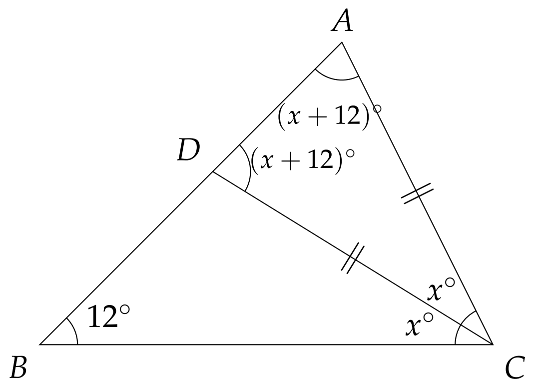
$$\begin{aligned}32^11 \cdot 125^{15} &= (2^5)^{11} \cdot (5^3)^{15} \\ &= 2^{55} \cdot 5^{45} \\ &= (2^{10} \cdot 2^{45}) \cdot 5^{45} \\ &= 2^{10} \cdot (2 \cdot 5)^{45} \\ &= 1024 \cdot 10^{45}.\end{aligned}$$

The number consists of the digits "1024" followed by 45 0's, and has $4 + 45 = 49$ digits.

3. In $\triangle ABC$ shown below, $\angle B = 12^\circ$, $\overline{CD}$ is the angle bisector of $\angle C$, and $CA = CD$. What is the degree measure of $\angle A$?




**Answer (64):** Let $\angle BCD = x^\circ$. Then $\angle ACD = x^\circ$ and $\angle CDA = (x + 12)^\circ$ since $\angle CDA$ is an exterior angle of $\triangle BCD$ whose measure equals the sum of the measures of interior angles $\angle B$ and $\angle DCB$. Moreover, $CA = CD$ implies $\angle A = m\angle CDA = (x + 12)^\circ$.




The interior angles of $\triangle ADC$ add up to $180^\circ$. Hence,

$$\begin{aligned}(x + 12) + (x + 12) + x &= 180\\ 3x + 24 &= 180\\ x &= 52.\end{aligned}$$

Then $\angle A = (x + 12)^\circ = (52 + 12)^\circ = 64^\circ$.

4. What is the smallest positive integer $n$ such that $12! \times n$ is the square of an integer?

   **Answer (**231**):** Note that $12! = 1 \times 2 \times 3 \times \ldots \times 12$. We can either find the prime factorization of 12!, or find groups of numbers which multiply to perfect squares, such as $(1, 4, 9)$, $(2, 3, 6)$, and $(5, 8, 10)$ in the product 12!. Therefore

$$\begin{aligned} 12! &= (1 \times 4 \times 9) \times (2 \times 3 \times 6) \times (5 \times 8 \times 10) \times 7 \times 11 \times 12 \\ &= 6^2 \times 6^2 \times 20^2 \times 7 \times 11 \times 2^2 \times 3 \\ &= K^2 \times 7 \times 11 \times 3 \end{aligned}$$

   where $K$ is an integer, equal to $2 \times 6 \times 6 \times 20)$ (the actual value of $K$ is irrelevant). Therefore, 12! must be multiplied by 3, 7, and 11 in order for the resulting product to be a perfect square, and we conclude that $n = 3 \times 7 \times 11 = 231$.

5. When a positive integer $A$ is divided by 50, the quotient is $n$ and the remainder is $n^2$ for some positive integer $n$. What is the maximum value of $A$?

   **Answer (**399**):** By definition, the remainder is less than the number that we are dividing by (in this case, 50). Therefore, $n^2 < 50$, implying $n \leq 7$. From the problem statement, we have $A = 50n + n^2$. To maximize $A$, we should make $n$ as large as possible. Setting $n = 7$ achieves the largest possible value of $A$, which is $A = 50(7) + 7^2 = 399$.

6. The radius of each rear wheel of a tractor is double the radius of each front wheel. After the tractor moved $960\pi$ inches without slippage, each front wheel made 20 more revolutions than each rear wheel. What is the diameter, in inches, of each rear wheel?

   **Answer (**48**):** Let the front and rear wheels of the tractor have radii $r$ and $2r$ inches, respectively. The diameter of each rear wheel is $2(2r) = 4r$. Suppose each rear wheel made $m$ revolutions. Then each front wheel made $m + 20$ revolutions. The total distance each wheel traveled is the same. Therefore, we can set up an equation involving the circumferences of the wheels:

$$\begin{aligned} 2\pi r(m + 20) &= 2\pi(2r)(m) \\ m + 20 &= 2m \\ m &= 20. \end{aligned}$$

   Therefore, the rear wheels made 20 revolutions and the front wheels made 40 revolutions. We can now solve for $r$ using the fact that the tractor moved $960\pi$

inches:

$$960\pi = 2\pi(2r)(20)$$
$$r = 12.$$

The diameter of each rear wheel is $4r = 4(12) = 48$ inches.

7. In a group of people, the average height of the males is 174 centimeters, the average height of the females is 164 centimeters, and the average height of all people in the group is 173.9 centimeters. What is the ratio of the number of males to the number of females in the group?

   **Answer (99):** Assume that there are $m$ males and $f$ females in this group. The total height of the males is $174m$ and the total height of the females is $164f$. The average height of the entire group is 173.9, which can also be computed in terms of $m$ and $f$:

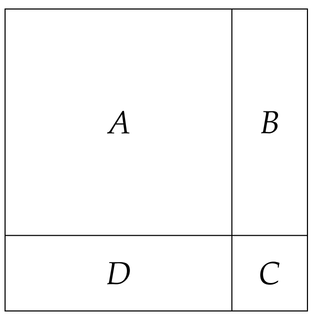
$$\frac{174m + 164f}{m + f} = 173.9.$$

   Cross-multiplying,

$$174m + 164f = 173.9m + 173.9f$$
$$0.1m = 9.9f,$$
$$\frac{m}{f} = \frac{9.9}{0.1} = 99.$$

   The desired ratio of males to females is 99:1, or 99.

8. In the figure below, a square is divided into two squares ($A$ and $C$) and two non-square rectangles ($B$ and $D$). Both diagonals of $B$ have length 16, and both diagonals of the large square have length 30. What is the sum of the areas of rectangles $B$ and $D$?



**Answer (194):** Let $a$ and $c$ be the side lengths of squares $A$ and $C$, respectively. The length of a diagonal of rectangle $B$ is $\sqrt{a^2+c^2}$ from the Pythagorean theorem, and the diagonal of the large square has length $(a+c)\sqrt{2}=30$. We have two equations relating $a$ and $c$:

$$a^2+c^2=16^2=256$$
$$2(a+c)^2=30^2=900.$$

Divide the second equation by 2 to obtain $(a+c)^2=450$. As $(a+c)^2=a^2+2ac+c^2$, we may substitute $a^2+c^2$ with 256 to obtain

$$2ac+256=450 \implies 2ac=194.$$

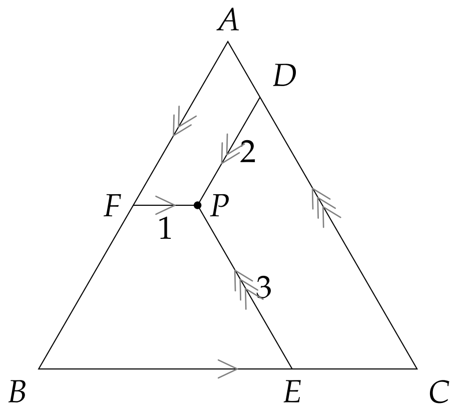
Rectangles $B$ and $D$ both have dimensions $a\times c$, so the sum of their areas is $2ac=194$.

9. The letters in the phrase "MAMMA MIA" are rearranged to make an eight-letter "word." In how many of these words does the first M appear before the first A? One example is MMAMAAIM.

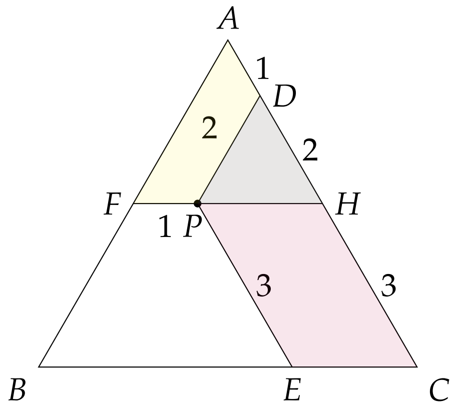
   **Answer (160):** Consider the simpler problem of rearranging four M's and three A's such that the first letter is an M. By setting the first M aside, this is equivalent to finding the number of rearrangements of three M's and three A's, in which there are $\binom{6}{3}=20$ ways.

   Next, given any of these 20 rearrangements, there are eight ways to place the "I." As the two problems of rearranging the letters AAAMMMM and inserting an I in this rearrangement are independent, the number of ways is $\binom{6}{3}\times 8=20\times 8=160$.

10. In the figure below, $\triangle ABC$ is an equilateral triangle. Segments $\overline{PD}$, $\overline{PE}$ and $\overline{PF}$ are parallel to sides $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$, respectively, and $PD=2$, $PE=3$, and $PF=1$. What is the square of the area of $\triangle ABC$?




**Answer (243):**




Extend $\overline{FP}$ so that it intersects $\overline{AC}$ at point $H$. Due to parallel lines $DP$ and $AB$, triangles $DHP$ and $AHF$ are equilateral. Hence, $DH = 2$ and $AD = 1$. Moreover, quadrilateral $PHCE$ has two pairs of parallel edges, so it is a parallelogram. Therefore, $HC = PE = 3$, giving $AC = 3 + 2 + 1 = 6$.

Recall that the area of an equilateral triangle of side length $s$ is $\frac{s^2\sqrt{3}}{4}$. Substituting $s = 6$, the area $K$ of $\triangle ABC$ is $K = \frac{6^2\sqrt{3}}{4} = 9\sqrt{3}$, and the desired answer is $K^2 = (9\sqrt{3})^2 = 243$.

11. Alice writes a sequence of digits on the blackboard. She realizes that any positive integer less than 1000 can be obtained by erasing some of the digits on the board. What is the fewest number of digits Alice could have written?

**Answer (29):** Let $S$ be the sequence of digits that Alice writes. Note that $S$ must include at least 3 of each digit from 1 to 9 and at least 2 zeros, because

$AAA$ and $A00$ must be attainable by erasing some digits of $S$ for any non-zero digit $A$. Hence, $S$ must have at least $3 \times 9 + 2 = 29$ digits. We can choose $S$ to be the following sequence of digits:

$$S = 123456789\,0123456789\,0123456789.$$

Any three-digit integer $\overline{ABC}$ can be obtained as follows: out of the first 9 digits, erase all digits except for $A$. Out of the next 10 digits, erase all except for $B$, and out of the last 10 digits, erase all except for $C$. Similarly, any 2-digit and 1-digit integer can also be obtained. Therefore, the fewest number of digits Alice could have written is 29.

12. How many positive 5-digit integers are there whose digits multiply to 80?

**Answer (155):** Since $80 = 2^4 \cdot 5$, the digits must be $(2,2,2,2,5)$, $(1,2,2,4,5)$, $(1,1,4,4,5)$, or $(1,1,2,8,5)$, up to ordering. Hence, there are $\frac{5!}{4!\,1!}$, $\frac{5!}{2!\,1!\,1!\,1!}$, $\frac{5!}{2!\,2!\,1!}$, and $\frac{5!}{2!\,1!\,1!\,1!}$ distinct integers that can be formed, for each of these cases. The desired answer is

$$\frac{5!}{4!\,1!} + \frac{5!}{2!\,1!\,1!\,1!} + \frac{5!}{2!\,2!\,1!} + \frac{5!}{2!\,1!\,1!\,1!} = 5 + 60 + 30 + 60 = 155.$$

13. The positive integer $N$ is such that the number $N^2$ is equal to the 4-digit number $\overline{AABB}$, where $A$ and $B$ are digits and $A \neq 0$. What is the value of $N$?

**Answer (88):** Note that

$$N^2 = \overline{AABB} = 1100A + 11B = 11(100A + B).$$

Therefore $N$ must be a multiple of 11, and $N^2$ must be a multiple of $11^2$. It follows that $100A + B$ must be a multiple of 11. We rewrite $100A + B$ as follows:

$$100A + B = 99A + A + B = 11(9A) + A + B.$$

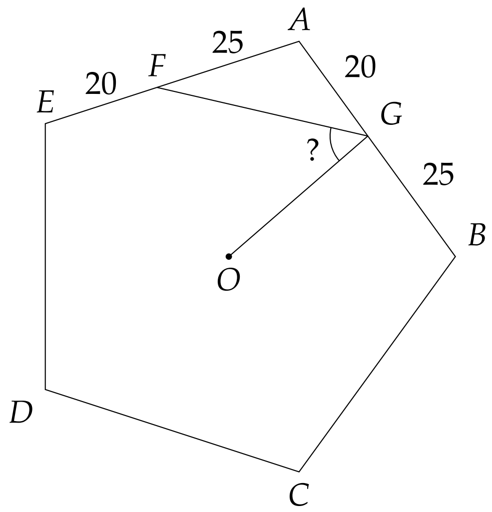
Equivalently, $A + B$ must be a multiple of 11. Since $A$ and $B$ are digits and $A \neq 0$, the only possibility is that $A + B = 11$.

Let $N = 11k$ for some positive integer $k$. Then

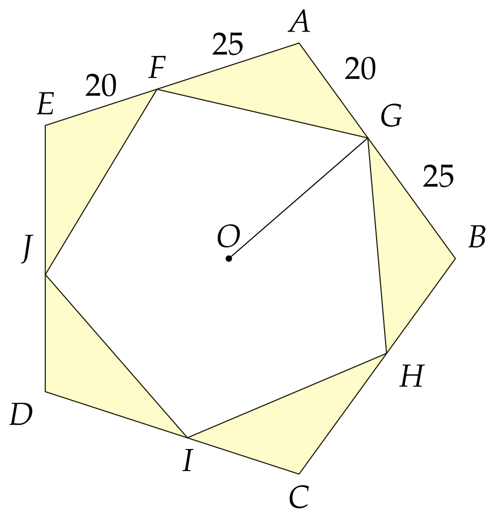
$$\begin{aligned} N^2 &= 11(100A + B) \\ (11k)^2 &= 11(99A + A + B) \\ 121k^2 &= 11(99A + 11) \\ 121k^2 &= 11^2(9A + 1) \\ k^2 &= 9A + 1. \end{aligned}$$

The only nonzero digit $A$ such that $9A+1$ is a perfect square is $A=7$, implying $k=8$. Then $N=11(8)=88$, and $N^2=7744$; the desired answer is 88.

14. $ABCDE$ is a regular pentagon and $O$ is its center. Points $F$ and $G$ are on $\overline{AE}$ and $\overline{AB}$, respectively, such that $AF=BG=25$ and $EF=AG=20$. What is the degree measure of $\angle OGF$?




**Answer (54):**




Pick the points $H$, $I$, and $J$ on $\overline{BC}$, $\overline{CD}$, and $\overline{DE}$, respectively, such that each point divides its respective side in the ratio $20:25$. Then all yellow triangles in the figure above are congruent due to side-angle-side (SAS) congruence.

This implies $FG = GH = HI = IJ = JF$, and that pentagon $FGHIJ$ is equilateral. Moreover, using the fact that the yellow triangles are congruent and $\angle A = \angle B = \ldots = \angle E = 108^\circ$, it can also be shown that the interior angles in $FGHIJ$ are also $108^\circ$. Therefore, $FGHIJ$ is a regular pentagon with center $O$. Since $\overline{OG}$ is an angle bisector, we conclude that

$$\angle OGF = \frac{1}{2}\angle FGH = \frac{108^\circ}{2} = 54^\circ.$$

15. 1000 eggs are distributed into 10 baskets, with each basket containing a different number of eggs. If the basket with the greatest number of eggs contains $N$ eggs, what is the smallest possible value of $N$?

    **Answer (105):** Suppose the basket with the greatest number of eggs contains $N$ eggs. Intuitively, we would like the numbers of eggs in the baskets to be as close as possible in order to minimize $N$. Suppose that the baskets contained $N-9, N-8, \ldots, N$ eggs. Since the baskets contain 1000 eggs altogether, we can consider the equation

    $$(N-9) + (N-8) + (N-7) + \ldots + N = 1000$$

    Simplifying and solving for $N$ yields

    $$\begin{aligned} 10N - 45 &= 1000 \\ N &= 104.5. \end{aligned}$$

    This is not an integer, implying that our initial assumption that we could place $N-9, N-8, \ldots, N$ eggs in the baskets was false. However, this does suggest that $N \approx 104.5$. If $N = 105$, then it is possible to distribute the eggs in this way, e.g., with 91, 97, 98, 99, ..., 104, 105 eggs. On the other hand, if $N \leq 104$, the total number of eggs is at most $95 + 96 + 97 + \ldots + 104 = 995 < 1000$, which is a contradiction. We conclude that the smallest possible value of $N$ is 105.

16. The following increasing sequence contains all positive integers which are equal to the sum of two distinct powers of 2:

    $$3, 5, 6, 9, 10, 12, 17, \ldots.$$

    The 2025$^{\text{th}}$ number in this sequence is equal to $2^a + 2^b$, where $a$ and $b$ are distinct positive integers. What is the value of $ab$?

    **Answer (512):** Consider the base 2 (binary) representations of the numbers in this sequence. Observe that each such number has exactly two 1's in the base 2 representation, and each number whose base 2 representation consist of exactly two 1's is counted. The number of such numbers having at most $n$ digits in binary representation is $\binom{n}{2}$, since it is equivalent to choosing two of the positions to place 1's in. We will first find $n$ such that

$$\binom{n}{2} \leq 2025 < \binom{n+1}{2}.$$

or equivalently, find the largest positive integer $n$ for which

$$\frac{n(n-1)}{2} \leq 2025.$$

This integer is $n = 64$, as $\binom{64}{2} = 2016 \leq 2025$ while $\binom{65}{2} > 2025$. This means that there are 2016 integers whose base 2 representations contain exactly two 1's out of the 64 digits whose place values are $2^{63}, 2^{62}, \ldots, 2^0$.

This means that the $2017^{\text{th}}$ number in the sequence will be $2^{64} + 2^0$, the $2018^{\text{th}}$ number will be $2^{64} + 2^1$, and so on. The $2025^{\text{th}}$ number will be $2^{64} + 2^8$, so $\{a, b\} = \{8, 64\}$ and $ab = 8 \times 64 = 512$.

17. How many positive integers less than or equal to 1000 are divisors of the number $13^{1000!} - 11^{1000!}$?

**Answer (**840**):** Note that 13 and 11 are not divisors of $13^{1000!} - 11^{1000!}$, nor are any multiples of 13 and 11. This follows as

$$\begin{aligned} \gcd(13, 13^{1000!} - 11^{1000!}) &= \gcd(13, 11^{1000!}) = 1, \text{ and} \\ \gcd(11, 13^{1000!} - 11^{1000!}) &= \gcd(11, 13^{1000!}) = 1. \end{aligned}$$

Hence, any divisor of $13^{1000!} - 11^{1000!}$ must be relatively prime to both 11 and 13. We claim that if $1 \leq d \leq 1000$ and $\gcd(d, 11) = \gcd(d, 13) = 1$, then $d$ is a divisor of $13^{1000!} - 11^{1000!}$. From Euler's totient theorem,

$$\begin{aligned} 13^{\varphi(d)} &\equiv 1 \pmod{d}, \\ 11^{\varphi(d)} &\equiv 1 \pmod{d}, \end{aligned}$$

where $\varphi(d)$ is the totient function; that is, the number of integers less than or equal to $d$ which are relatively prime with $d$. Since $\phi(d) \leq d \leq 1000$, it follows that $\phi(d)$ is a factor of 1000!, which implies

$$13^{1000!} \equiv 11^{1000!} \equiv 1 \pmod{d}$$

or equivalently, $13^{1000!} - 11^{1000!} \equiv 0 \pmod{d}$, as claimed. Therefore we only need to find the number of positive integers $d \in \{1, 2, 3, \ldots, 1000\}$ which are not divisible by 11 or 13. Note that $1001 = 7 \times 11 \times 13$ is divisible by 11 and 13. Out of the first 1001 positive integers, exactly $\frac{10}{11}$ of them are relatively prime to 11, and among these numbers, $\frac{12}{13}$ of them are relatively prime to 13. The number of integers $d$ is

$$1001 \times \frac{10}{11} \times \frac{12}{13} = 7 \times 12 \times 10 = 840.$$

Note that this does not include 1001, so we can conclude that the answer is 840.

18. A positive divisor of $6^{1000}$ is chosen at random. If the probability that the chosen number is a multiple of $6^{91}$ is $\dfrac{m}{n}$ where $m$ and $n$ are relatively prime positive integers, what is $m+n$?

**Answer (221):** Any divisor of $6^{1000}$ is of the form $2^a 3^b$ where $a$ and $b$ are integers such that $0 \leq a \leq 1000$ and $0 \leq b \leq 1000$. Among these divisors, the multiples of $6^{91}$ are of the form $2^A 3^B$ where $A$ and $B$ are integers such that $91 \leq A \leq 1000$ and $91 \leq B \leq 1000$. There are 1001 choices for $a$ and $b$, and $1001^2$ divisors of $6^{1000}$. Of these divisors, there are $1000 - 91 = 910$ ways to choose $A$ and $B$, and $910^2$ divisors which are multiples of $6^{91}$. The desired probability $p$ is

$$p = \frac{910 \times 910}{1001 \times 1001} = \frac{10^2}{11^2} = \frac{100}{121}.$$

Then $m + n = 100 + 121 = 221$.

19. Let $P(x)$ be a quadratic polynomial such that $P(1) = 2$, $P(2) = 3$, and $P(3) = 1$. What is the value of $P(4)^2$?

**Answer (16):** Let $P(x) = ax^2 + bx + c$ where $a$, $b$, and $c$ are coefficients. Using the given information, we can form a system of equations:

$$a + b + c = 2 \quad (1)$$
$$4a + 2b + c = 3 \quad (2)$$
$$9a + 3b + c = 1. \quad (3)$$

When (1) is subtracted from (2), and (2) is subtracted from (3), we get two equations in $a$ and $b$:

$$3a + b = 1$$
$$5a + b = -2.$$

Solving the above system using any method yields $(a, b) = (-\frac{3}{2}, \frac{11}{2})$, and substituting these values of $a$ and $b$ into any equation implies $c = -2$. Therefore,

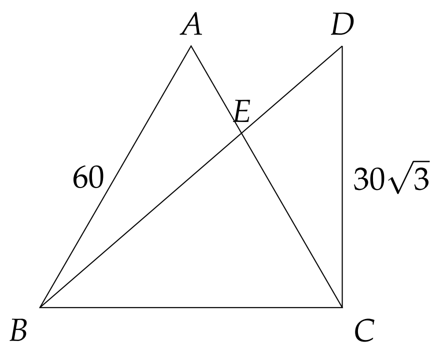
$$P(x) = -\frac{3}{2}x^2 + \frac{11}{2}x - 2.$$

Substituting $x = 4$, we find that $P(4) = -4$, and $P(4)^2 = (-4)^2 = 16$.

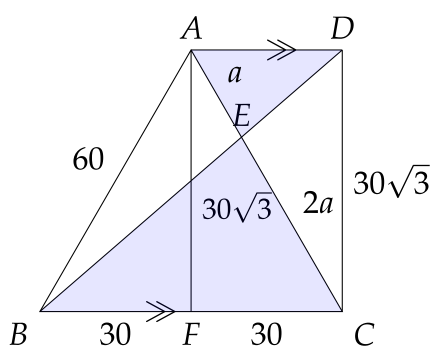
**Alternate Solution:** We can use the *finite differences* method to find $P(4) = -4$ given $P(3)$, $P(2)$, and $P(1)$. Since $P(x)$ is quadratic, the second finite differences (denoted $\Delta^2$) must be constant:

| | $P(1)$ | | $P(2)$ | | $P(3)$ | | $P(4)$ |
|---|---|---|---|---|---|---|---|
| $\Delta^0$ | 2 | | 3 | | 1 | | $\mathbf{-4}$ |
| $\Delta^1$ | | 1 | | $-2$ | | $\mathbf{-5}$ | |
| $\Delta^2$ | | | $-3$ | | $\mathbf{-3}$ | | |

20. In the figure below, $\triangle ABC$ is an equilateral triangle, $AB = 60$, $CD = 30\sqrt{3}$, and $\angle BCD$ is a right angle. What is $AE^2$?




**Answer (400):**




Let $F$ be the foot of altitude from $A$ to $\overline{BC}$. Observe that $AF = 30\sqrt{3} = DC$. Since $\angle F = \angle C = 90^\circ$, $AFCD$ is a rectangle, implying $AD = 30$ and $AD \parallel BC$. Triangles $ADE$ and $CBE$ are similar due to parallel lines and AA similarity, and the similarity ratio is $30 : 60 = 1 : 2$. If we set $AE = a$, then $EC = 2a$. On the other hand, $AC = 60 = 3a$; hence $a = 20$, and $AE^2 = 20^2 = 400$.

21. Let the sequence $a_n$ be defined for all positive integers $n$ by the rule

$$a_n = n^2 + n + 1.$$

Given that $a_{30}a_{31} = a_N$ for some integer $N$, what is the value of $N$?

**Answer (**961**):** Let us try some small cases:

$$\begin{aligned} a_1a_2 &= 3(7) = 21 = a_4 \\ a_2a_3 &= 7(13) = 91 = a_9 \\ a_3a_4 &= 13(21) = 273 = a_{16} \\ a_4a_5 &= 21(31) = 651 = a_{25}. \end{aligned}$$

We can immediately conjecture that $a_na_{n+1} = a_{(n+1)^2}$; substituting $n = 30$ yields $a_{30}a_{31} = a_{961}$, and $N = 961$.

**Remark:** To prove the above conjecture, we can compute $a_na_{n+1}$ and $a_{(n+1)^2}$ in terms of $n$, and show that they are equal:

$$\begin{aligned} a_na_{n+1} &= (n^2 + n + 1)[(n+1)^2 + (n+1) + 1] \\ &= (n^2 + n + 1)(n^2 + 3n + 3) \\ &= n^4 + 4n^3 + 7n^2 + 6n + 3 \end{aligned}$$

On the other hand,

$$\begin{aligned} a_{(n+1)^2} &= ((n+1)^2)^2 + (n+1)^2 + 1 \\ &= (n^4 + 4n^3 + 6n^2 + 4n + 1) + (n^2 + 2n + 1) + 1 \\ &= n^4 + 4n^3 + 7n^2 + 6n + 3. \end{aligned}$$

These two expressions are equal; therefore, $a_na_{n+1} = a_{(n+1)^2}$.

22. In a 25-question test, each correct answer earns 4 points, leaving a question blank earns 0 points, and each incorrect answer incurs a deduction of 1 point. How many different scores are possible?

**Answer (**120**):** Suppose a student answers $x$ questions correctly and $y$ questions incorrectly, where $x + y \leq 25$. Then the total score is $4x - y$. The minimum score is $-25$ and the maximum score is 100, so there are 126 possible scores to check.

Observe that the 26 scores $-25, -24, \cdots, -1, 0$ are attainable even if $x = 0$ (e.g., by answering 0 correctly, and answering $y$ incorrectly). Additionally, every positive score $s$ less than or equal to $88 = 22 \times 4$ is attainable by answering $\lceil \frac{s}{4} \rceil$ questions correctly (necessarily between 1 and 22, inclusive), and either 1, 2, or 3 questions incorrectly. This gives 88 possible scores.

We can manually check the cases where $x = 23$ and $0 \le y \le 2$, where $x = 24$ and $0 \le y \le 1$, or where $x = 25$. This gives 6 additional scores: 90, 91, 92, 95, 96, and 100. Altogether, there are $26 + 88 + 6 = 120$ possible scores.

23. The numbers in the set $\{1, 2, 3, 4, \ldots, 75\}$ are divided into 15 subsets with each subset containing exactly 5 numbers. No two subsets have a number in common. What is the largest possible value of the sum of the medians of the subsets?

**Answer (780):** Denote the subsets $S_1 = \{a_1, b_1, c_1, d_1, e_1\}$, $S_2 = \{a_2, b_2, c_2, d_2, e_2\}$, $\ldots$, $S_{15} = a_{15}, b_{15}, c_{15}, d_{15}, e_{15}\}$, where $a_i < b_i < c_i < d_i < e_i$ for all $i \in \{1, \ldots, 15\}$. The problem is to maximize $c_1 + c_2 + \ldots + c_{15}$.

Note that since we want the medians as large as possible, we can greedily try to maximize the sum of the medians by choosing the sets as follows:

$$\begin{aligned} S_1 &= \{29, 30, 73, 74, 75\} \\ S_2 &= \{27, 28, 70, 71, 72\} \\ S_3 &= \{25, 26, 67, 68, 69\} \\ &\vdots \\ S_{15} &= \{1, 2, 31, 32, 33\}. \end{aligned}$$

For this choice of sets, the sum of the medians is $31 + 34 + 37 + \ldots + 73 = 780$.

We will now prove that 780 is the largest possible sum. To do this, observe that $c_i \le d_i - 1$ and $c_i \le e_i - 2$ for all $i$, since $c_i < d_i < e_i$. Equivalently, $c_i + 1 \le d_i$ and $c_i + 2 \le e_i$. We will consider the sum $\sum_{i=1}^{15} (c_i + d_i + e_i)$, which is upper bounded by the sum of the 45 largest numbers, namely $31 + 32 + 33 + \ldots + 75$:

$$\begin{aligned} \sum_{i=1}^{15} c_i + (c_i + 1) + (c_i + 2) &\le \sum_{i=1}^{15} (c_i + d_i + e_i) \\ &\le 31 + 32 + 33 + \ldots + 75 \\ &= 2385. \end{aligned}$$

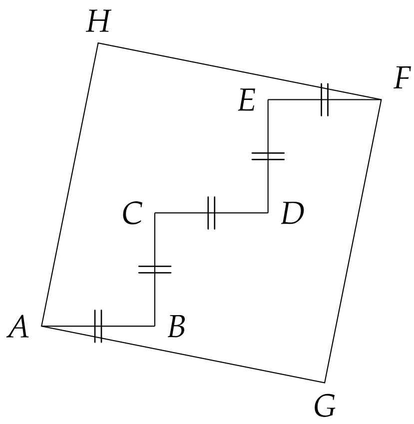
The sum on the left-hand side can also be written as $\sum_{i=1}^{15} (3c_i + 3) = 3 \left( \sum_{i=1}^{15} c_i \right) +$

45. It follows that

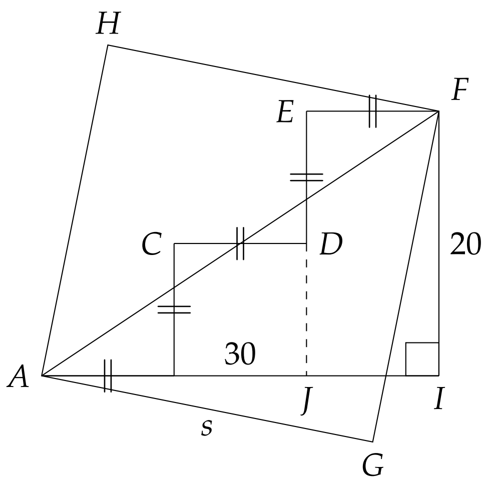
$$3\left(\sum_{i=1}^{15} c_i\right) + 45 \leq 2385$$
$$\sum_{i=1}^{15} c_i \leq \frac{2385 - 45}{3} = 780.$$

This shows that the sum of the medians is upper bounded by 780. Since we provided an example $(S_1, S_2, \ldots, S_{15})$ whose sum of medians is 780, we conclude that 780 is the largest possible sum.

24. In the figure below, $AGFH$ is a square, the angles at $B$, $C$, $D$ and $E$ are all right angles, and $AB = BC = CD = DE = EF = 10$. What is the area of square $AGFH$?




**Answer (650):** Let $I$ be the foot of the perpendicular from $F$ onto the extension of $\overline{AB}$:



Since $CBJD$ is a square and $EJIF$ is a rectangle, $AI = AB + BJ + JI = 30$, and $FI = EJ = ED + DJ = 20$. Let one side of square $AGFH$ be $s$. The length of diagonal $\overline{AF}$ is $s\sqrt{2}$. By considering $\triangle AIF$, the length of hypotenuse $\overline{AF}$ is also $\sqrt{30^2 + 20^2} = \sqrt{1300}$. It follows that $s\sqrt{2} = \sqrt{1300} \implies 2s^2 = 1300 \implies s^2 = 650$. Since $s^2$ represents the area of square $AGFH$, we conclude that its area is 650.

25. Real numbers $x$ and $y$ satisfy the equation $x^2 - y^2 = 100$. What is the smallest possible value of the expression $(3x + y)^2$?

**Answer (**800**):** While this can be solved using calculus, we show a solution without calculus. We can rewrite the given equation using difference of squares:

$$(x - y)(x + y) = 100.$$

The key idea is to rewrite $3x + y$ in terms of $x - y$ and $x + y$:

$$3x + y = 2(x + y) + (x - y).$$

Define $A = x + y$ and $B = x - y$. Then we can equivalently minimize $(2A + B)^2$, given that $AB = 100$. We have

$$(2A + B)^2 = (2A - B)^2 + 8AB \geq 8AB = 800,$$

and the equality case holds when $2A - B = 0$, or $2A = B$. The minimum value is 800, which holds when $A = \pm 5\sqrt{2}$ and $B = \pm 10\sqrt{2}$, or $(x, y) = (\pm\frac{15\sqrt{2}}{2}, \mp\frac{5\sqrt{2}}{2})$. Note that the values $x$ and $y$ which minimize are not needed for this problem; it is only necessary to show that these values exist in order to show that the minimum value is 800.