Pi Math Contest Gauss Division

2024

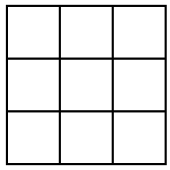
INSTRUCTIONS

1. DO **NOT** OPEN THIS BOOKLET UNTIL YOUR PROCTOR TELLS YOU!
2. This is a 25 question test. Each question has a three digit answer: 000, 001, 002, 003, . . ., 999. For example, if the answer is 1, you must bubble 001 instead of 1.
3. Mark your answer to each question on the Answer Form with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded.
4. SCORING: You will receive 10 points for each correct answer, 1 point for each problem left unanswered, and 0 points for each incorrect answer.
5. Only pencils, pencil sharpeners, erasers, and blank scratch papers are allowed. All other aids, including but not limited to calculators and notes, are not allowed.
6. Figures are not necessarily drawn to scale.
7. When your proctor gives the signal, begin working on the problems. You will have **75 minutes** to complete the test.
8. After the exam, return your **Answer Form, Test Booklet and scratch papers** to your proctor.
9. Do NOT discuss any exam questions until **March 18th**, after which problems and solutions will be available on the contest website.

1. The expression
$$\frac{1}{\frac{1}{20}+\frac{1}{24}}$$
equals $\frac{a}{b}$ in simplest form, where a and b are relatively prime positive integers. What is $a+b$?

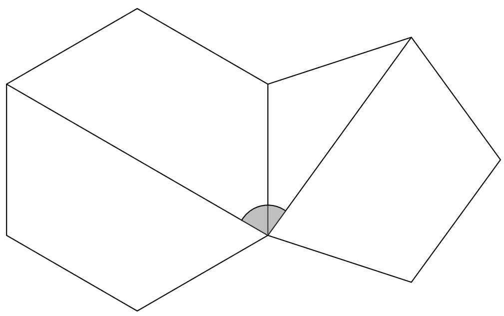
2. If it takes 540 seconds to cut a log into 19 pieces, how many seconds are needed to cut it into 7 pieces at the same pace? The cutting is done one piece at a time.

3. A 3×3 metal grid, as shown below, is created using a total of 120 inches of metal wire. Determine the area of the square grid.



4. When the 4-digit number $\underline{PIMC}$ is divided by the 3-digit number $\underline{PIM}$, the quotient and the remainder add up to a perfect square. What is the value of C?

5. A regular hexagon and a regular pentagon, both with sides of the same length, are attached to each other along one of their sides, as shown below. Determine the degree measure of the angle indicated in the diagram.



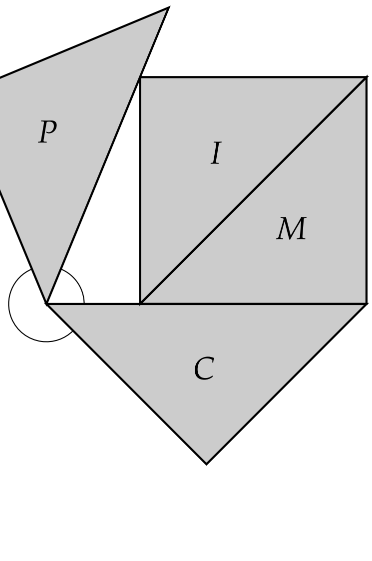
6. Six friends, including Maxy and Lily, seat themselves in a row of six adjacent seats at the movies. How many arrangements are possible where there are an odd number of seats between Maxy and Lily?

7. Twelve years ago, a father was three times as old as his son. Eight years from now, he will be twice as old as his son. What is the sum of the ages of the father and son now?

8. Let A, B, and C be distinct positive digits. When these digits are arranged to form the 3-digit numbers $\underline{ABC}$, $\underline{CAB}$, and $\underline{BCA}$, it is observed that $\underline{ABC}$ is divisible by 4, $\underline{CAB}$ is divisible by 5, and $\underline{BCA}$ is divisible by 9. What is the value of the largest of these 3-digit numbers?

9. A family with two parents and six teenager children rides in an eight-seat SUV, featuring one row of two seats (including the driver's seat) and two rows with three seats each. One of the parents must occupy the driver's seat. Three of the children are triplets and wish to be seated in the same row. How many ways can the family seat themselves, subject to these two conditions?

10. Let P, I, M and C be congruent isosceles right triangles arranged on a plane, as shown in the figure below. What is the degree measure of the positive difference between the larger and smaller angles between triangles P and C?

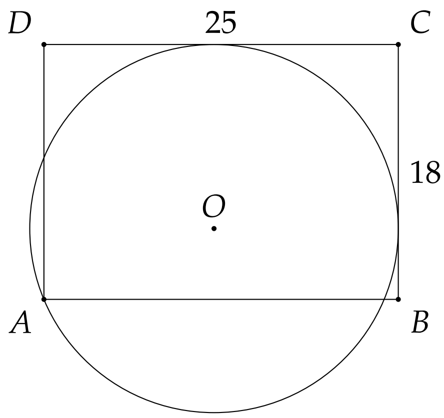


11. A cashier has an ample supply of \$1, \$5, \$10, \$20, and \$50 bills. For some positive integer d, it is impossible to make \$$d$ using fewer than ten of these bills. What is the smallest possible value of d?

12. For how many of the first 1000 positive integers n is the non-negative difference between the decimal parts of $\frac{n}{2}$ and $\frac{n}{3}$ equal to $\frac{1}{6}$? (The *decimal part* of a number equals the part after the decimal point; for example, the decimal part of 3.5 is 0.5.)

13. What is the area, in square units, of the region in the xy-plane satisfying the inequality $\lfloor |x| \rfloor + \lfloor |y| \rfloor \leq 3$? The notations $|r|$ and $\lfloor r \rfloor$ respectively denote the absolute value of r and the greatest positive integer less than or equal to r.

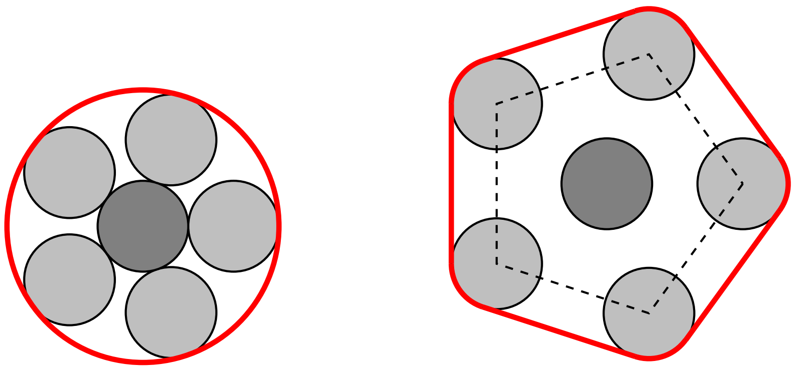
14. In rectangle $ABCD$ below, $CB = 18$ and $CD = 25$. A circle centered at O passes through point A and is tangent to sides $\overline{CB}$ and $\overline{CD}$. What is OD^2?



15. Nine chess players participate in a tournament and are ranked from 1 to 9. Given that Elizabeth placed 5th, Anna placed ahead of Benjamin, Benjamin placed ahead of Carly, Danielle placed ahead of Elizabeth, Elizabeth placed ahead of Fiona, and Gary placed ahead of both Harold and Isaac, how many possible rankings are there?

16. Three runners compete in a 400-meter race. When the first runner crosses the finish line, the second and third runners are 10 and 49 meters behind the finish line, respectively. When the second runner finishes the race, how many meters will the third runner be behind the finish line? Assume each runner runs at a constant speed.

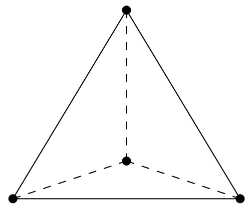
17. Six identical hockey pucks, each with a radius of 1 inch, are arranged as shown on the left below. Five of these pucks are equally spaced and touch the center puck. A circular rubber band is then placed around the outer five pucks. Upon moving the five outer pucks away from the center puck, the rubber band doubles in length as shown on the right below. Let P be the perimeter of the pentagon whose vertices are the centers of the pucks, as shown on the right below. What is the integer closest to P?



18. Amelia and Brandon take turns taking candies from a jar initially containing N candies: Amelia takes 1 candy, Brandon takes 2, Amelia takes 3, and so on, where the next person always takes one more candy. If there aren't enough candies in the jar, that person takes all remaining candies. If Amelia takes 2024 candies altogether, what are the last three digits of N?

19. How many distinct ways can the set $S = \{1, 2, 4, 8, \ldots, 2^9\}$ be partitioned into two disjoint, non-empty subsets A and B so that $A \cup B = S$, and the sum of the elements in A is a multiple of the sum of the elements in B? One example is $A = \{4, 8, \ldots, 2^9\}$ and $B = \{1, 2\}$.

20. Four ants are initially located at different vertices of a regular tetrahedron. Each ant randomly moves to one of the three adjacent vertices. If the probability that each ant ends up at a different vertex is $\frac{m}{n}$ where m and n are relatively prime integers, what is $m+n$?



21. Olivia selects a quadratic polynomial. She multiplies the quadratic polynomial by two different linear polynomials and displays the resulting products but erases the constant terms in both products. When Elijah enters the room, she sees the two expressions:

$$x^3 + 2x^2 - x$$

$$x^3 - x^2 - 4x$$

Despite the missing constant terms, Elijah can correctly determine the sum of the squares of the roots of Olivia's original quadratic polynomial. What is this sum?

22. How many integer pairs (a, b) are there such that $1 \le a \le 1000$, $1 \le b \le 1000$, and $\operatorname{lcm}(a, b) = 30 \gcd(a, b)$?

23. Isabella rolls a fair six-sided dice repeatedly until she rolls every number at least once. What is the closest integer to the expected number of rolls needed?

24. A real number $3 \le r \le 1000$ is chosen, and a sequence is constructed by the rules $a_0 = r$ and $a_n = \sqrt{15a_{n-1} - 36}$ for $n \ge 1$. What is the sum of all possible values of $\lfloor a_{2024} \rfloor$? The notation $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

25. Sophia has a collection of rocks with possibly different weights. She discovers that she can distribute these rocks into 3 piles of equal weight, 4 piles of equal weight, and also 5 piles of equal weight. What is the minimum positive number of rocks in her collection for which this is possible?