Pi Math Contest Fermat Division

2024 Solutions

The problems and solutions in this contest were proposed by:

Edwin Peng, Emre Gul, Hanna Chen, Isaac Li, Justin Sun, Kelly Cui, Michael Zhang, Preston Fu, Richard Spence, and Stanley Wang.

Solutions

1. Allie purchases three t-shirts, priced at \$13.50 each. Dana purchases two jackets, priced at \$27.75 each. How many more dollars does Dana spend than Allie?

   Answer (15): Allie spends $3 \times \$13.50 = \$40.50$ on t-shirts, and Dana spends $2 \times \$27.75 = \$55.50$ on jackets. Dana spends $\$55.50 - \$40.50 = \$15$ more than Allie.

2. What integer is equal to 36% of 75?

   Answer (27): 36% of 75 is equal to 75% of 36, which is $\frac{3}{4} \times 36 = 27$.

3. A recipe for 18 chocolate chip cookies calls for $1\dfrac{2}{3}$ cups of flour. If Abby has 5 cups of flour and an ample supply of every other ingredient, what is the greatest number of chocolate chip cookies Abby can make?

   Answer (54): Since $1\frac{2}{3} = \frac{5}{3}$, Abby can scale her recipe by a factor of 3 since she has 5 cups of sugar. The greatest number of cookies Abby can make is $18 \times 3 = 54$.

4. Hector is six years older than Marisol, Marisol is 4 years older than Cielo, and Hector is twice as old as Cielo. What is the sum of their ages?

   Answer (44): Let h, m, and c be the ages of Hector, Marisol, and Cielo, respectively. We know $h = m + 6$ and $m = c + 4$, so $h = c + 10$, and Hector is 10 years older than Cielo. Since Hector is twice as old as Cielo, $h = 2c$, so $2c = c + 10 \implies c = 10$. Then Cielo is 10, Marisol is 14, and Hector is 20, and the sum of their ages is $20 + 14 + 10 = 44$.

5. The greatest common divisor of n and 180 is 6. Given that n is a positive integer less than 100, what is the greatest possible value of n?

   Answer (78): The prime factorization of 180 is $2^2 \times 3^2 \times 5$. Therefore, in order for $\gcd(n, 180)$ to equal $2^1 \times 3^1 = 6$, the prime factorization of n must be $2^1 \times 3^1 \times k$, where k is not divisible by 2, 3, or 5 (if k was divisible by any of these prime numbers, then the GCD would not be 6). Since $n < 100$, the first possible candidate is $96 = 6 \times 16$. However, $k = 16$ in this case, and $\gcd(96, 180) \neq 6$. Similarly, $90 = 6 \times 15$ and $84 = 6 \times 14$ do not work, since 15 and 14 are divisible by 3 and 2, respectively. However, $78 = 6 \times 13$ works, since 13 is not divisible by 2, 3, or 5. Therefore, the greatest possible value of n is 78.

6. Ten people, including Robert and Saul, randomly seat themselves at a round table containing ten chairs. The probability that Robert and Saul do **not** sit next to each other is equal to $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

   Answer (16): Regardless of which chair Robert sits in, there are 9 seats left, of which 2 of them are next to Robert. The probability that Saul does not sit next to Robert is $\frac{7}{9}$, so $m + n = 7 + 9 = 16$.

7. Let x and y be positive integers such that $xy + x + y = 220$. What is $x + y$?

   Answer (28): After adding 1 to both sides, we can factor the left-hand side:
   $$xy + x + y + 1 = (x + 1)(y + 1) = 221.$$
   The prime factorization of 221 is 13×17; this can also be seen using the difference of squares method, as $221 = 15^2 - 2^2 = (15 - 2)(15 + 2)$. Since

x and y are positive integers, $x+1$ and $y+1$ are at least 2. Therefore $x+1 = 13$ and $y+1 = 17$ (or vice versa), implying $\{x, y\} = \{12, 16\}$. Then $x + y = 12 + 16 = 28$.

8. Parallelogram $ABCD$ has vertices $A\,(1,1)$, $B\,(2,5)$, $C\,(a,b)$ and $D\,(7,3)$ in the xy-plane. What is $a+b$?

   Answer (15): In parallelogram $ABCD$, the diagonals AC and BD bisect each other. Then the midpoint of AC is the same as the midpoint of BD, so
$$\left(\frac{1+a}{2}, \frac{1+b}{2}\right) = \left(\frac{2+7}{2}, \frac{5+3}{2}\right).$$
   It follows that $1+a = 2+7 = 9$ and $1+b = 5+3 = 8$, so $(a,b) = (8,7)$ and $a+b = 15$.

9. Eric and Joshua are sitting near a farm containing 2-legged chickens and 4-legged cows, and no other animals. Eric counts 102 legs, and Joshua counts 39 animals. How many chickens are on the farm?

   Answer (27): Let x denote the number of cows on the farm and y denote the number of chickens. Because Eric counts 102 legs, we have the equation $4x + 2y = 102$. Since Joshua counts 39 animals, we have $x + y = 39$. To find y, we can multiply the second equation by -4, then add the first equation, obtaining $(4x+2y) + (-4x-4y) = 102 - 4(39) = -54$, so $-2y = -54$ and $y = 27$.

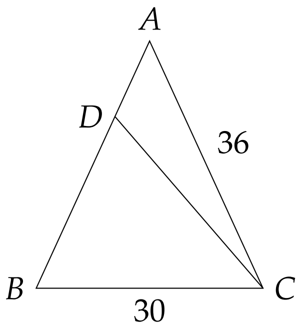
   Alternate Solution: If all 39 animals were chickens, there would be $2 \times 39 = 78$ legs. However, there are $102 - 78 = 24$ extra legs due to the cows. If one chicken was replaced with a cow, there would be 2 more legs. Therefore, there are $24 \div 2 = 12$ cows, and $39 - 12 = 27$ chickens.

10. Consider the statement "If a positive 2-digit integer is not divisible by 2, 3, or 5, then it is prime." How many positive 2-digit integers make this statement false?

    Answer (03): A counterexample to this statement would be a positive 2-digit integer which is not divisible by 2, 3, or 5, and is not prime (i.e., composite, as it must be greater than 9 and cannot equal 0 or 1). The composite

2-digit integers which are not divisible by 2, 3, or 5 can only be divisible by primes greater than 5. These integers are: $49 = 7 \times 7, 77 = 7 \times 11$, and $91 = 7 \times 13$. There are 3 integers.

11. Isosceles triangle ABC has $m\angle ABC = m\angle ACB$. Point D is on side AB so that $m\angle BDC = m\angle ABC$. Given that $AC = 36$ and $BC = 30$, what is the perimeter of triangle ACD?



Answer (77): Let $m\angle ABC = m\angle ACB = \alpha$. Since triangle ABC is isosceles, it follows that $AB = AC = 36$.

Moreover, since $m\angle BDC = m\angle ABC = \alpha$, it follows that $\triangle CBD$ is also isosceles, with $CD = BC = 30$. Moreover, $\triangle CBD \sim \triangle ABC$ by AAA similarity. Then $\frac{BD}{BC} = \frac{BC}{AB}$, or $\frac{BD}{30} = \frac{30}{36} \implies BD = 25$. Therefore, $AD = 36 - 25 = 11$.

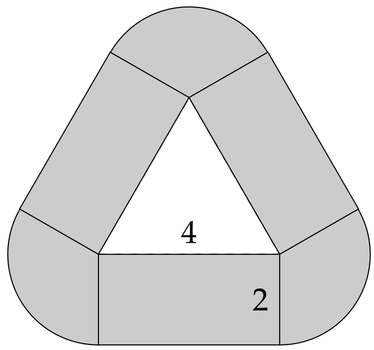
The perimeter of $\triangle ACD$ is $36 + 30 + 11 = 77$.

12. The positive integer a has exactly 5 positive factors, and the positive integer b has exactly 7 positive factors. The integer $a \times b$ must either have m positive factors or n positive factors, where $m \neq n$. What is $m + n$?

Answer (46): The only possibility is that $a = p^4$ for some prime number p, and $b = q^6$ for some prime number q. If $p = q$, then $a \times b = p^{10}$, and $a \times b$ has 11 factors. Otherwise, if $p \neq q$, then $a \times b = p^4 \times q^6$, which has $(4 + 1) \times (6 + 1) = 5 \times 7 = 35$ factors. Therefore $\{m, n\} = \{11, 35\}$, and $m + n = 11 + 35 = 46$.

13. An equilateral triangle with side length 4 cm is drawn on the plane. Let $\mathcal{S}$ be the set of points in the plane that are outside the triangle, but within 2 cm of some point on or inside the triangle. The area of $\mathcal{S}$ equals $a\pi + b$ cm^2 for some positive integers a and b. What is $a + b$?

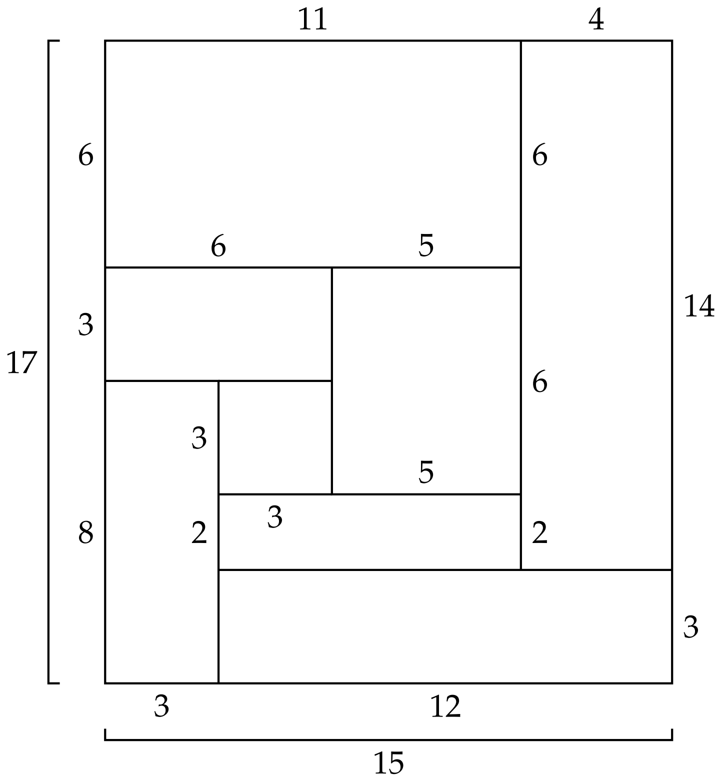
    Answer (28): The set $\mathcal{S}$ of points consists of three 4 cm $\times$ 2 cm rectangles, and three 120° sectors as shown in the figure below.



    The combined area of the three rectangles is $3 \times (4 \times 2) = 24$ cm^2, and the combined area of the three 120° sectors equals the area of a circle of radius 2 cm, which is $2^2\pi = 4\pi$ cm^2. The area of $\mathcal{S}$ is $4\pi + 24$ cm^2, so $a = 4$, $b = 24$, and $a + b = 4 + 24 = 28$.

14. Eight rectangles with dimensions 2×8, 3×3, 3×6, 3×8, 3×12, 4×14, 5×6, and 6×11 are arranged to form a larger rectangle, without any overlap or gaps. What is the perimeter of the larger rectangle?

    Answer (64): The area of the larger rectangle is the combined area of the rectangles, which is $16 + 9 + 18 + 24 + 36 + 56 + 30 + 66 = 255$. The prime factorization of 255 is $3 \times 5 \times 17$, so one of the side lengths must be a prime. Since the 6×11 is contained inside the rectangle, both dimensions must be greater than or equal to 6, and the only possibility is that the larger rectangle is a 15×17 rectangle. The perimeter of this rectangle is $2(15 + 17) = 64$. While not necessary to solve the problem, the figure below shows one way the rectangles can be arranged to form a 15×17 rectangle.



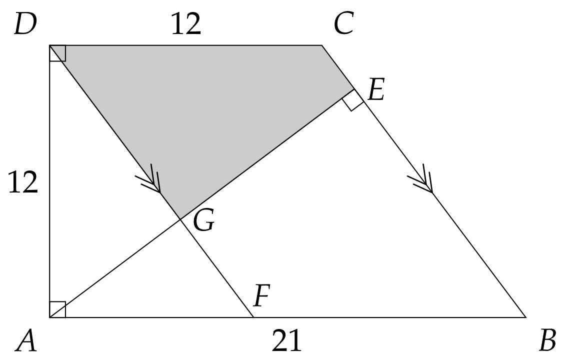
15. David cuts a 4-inch cube into 64 unit cubes, each with side length 1 inch. Half of these cubes are painted red on all faces, and the rest are painted blue on all faces. When David reassembles the unit cubes to form a 4-inch cube, what is the greatest possible surface area of this cube, in square inches, which is painted red?

    Answer (72): The $4'' \times 4'' \times 4''$ cube contains 8 "vertex" unit cubes (cubes with three faces exposed), 24 "edge" unit cubes (cubes with two faces exposed), and 32 remaining cubes (cubes with one or no faces exposed). To maximize the surface area which is painted red, the 8 vertex cubes and 24 edge cubes should be red. The greatest possible surface area which is painted red is $8 \times 3 + 24 \times 2 = 24 + 48 = 72$ square inches.

16. For how many integers n between 1 and 100, inclusive, is the product $n(n+1)(n+2)$ divisible by 15?

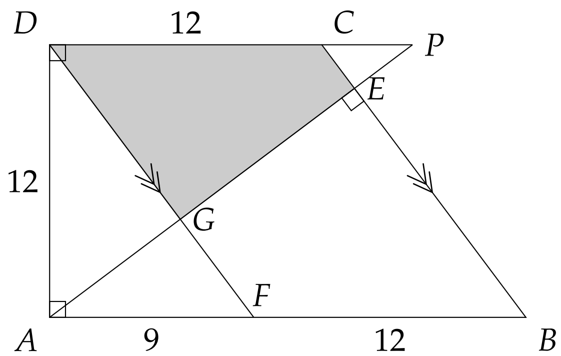
Answer (60): In order for the product $n(n+1)(n+2)$ to be divisible by 15, it must be divisible by 3 and 5. Note that $n(n+1)(n+2)$ is the product of three consecutive integers. Then one of these must be a multiple of 3, so $n(n+1)(n+2)$ is divisible by 3 regardless of n. In order for $n(n+1)(n+2)$ to be divisible by 5, n must have units digit 0, 3, 4, 5, 8, or 9. Then exactly $\frac{6}{10}$ of the first 100 positive integers will satisfy the property that $n(n+1)(n+2)$ is divisible by 15, so the answer is $\frac{6}{10} \times 100 = 60$.

17. Trapezoid $ABCD$ has $AB \parallel CD$, and $m\angle CDA = m\angle DAB = 90°$. Point E is on segment BC so that $AE \perp BC$, and point F is on AB so that $DF \parallel BC$. Segments AE and DF intersect at G. Given that $AB = 21$ and $CD = AD = 12$, what integer is closest to the area of trapezoid $CDGE$?



Answer (58): Since $BF = CD = 12$, it follows that $AF = 21 - 12 = 9$. Then $\triangle ADF$ is similar to a 3-4-5 right triangle.

Extend AE and CD to intersect at point P, as shown:



By a quick angle chase and AAA argument, we have that $\triangle GPD \sim \triangle ADF \sim \triangle DPA$, and all are similar to 3-4-5 triangles. Since $AD = 12$, it follows that $DP = \frac{4}{3} \times 12 = 16$, and $CP = 16 - 12 = 4$.

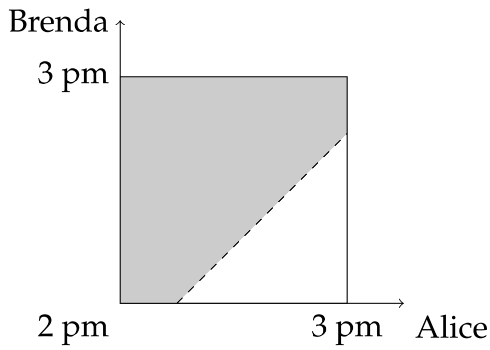
The area of trapezoid $CDGE$ equals the area of $\triangle GPD$, minus the area of right $\triangle EPC$. The base and height of $\triangle GPD$ are $DG = 16 \times \frac{3}{5} = \frac{48}{5}$, and $GP = 16 \times \frac{4}{5} = \frac{64}{5}$, respectively. The area of $\triangle GPD$ is $\frac{1}{2} \times \frac{48}{5} \times \frac{64}{5} = \frac{32\times 48}{25}$. Since $\triangle EPC$ is similar to $\triangle GPD$ with a scale factor of 1:4, the area of $\triangle EPC$ is $\frac{1}{16}$ the area of $\triangle GPD$, and therefore the area of $CDGE$ is

$$[CDGE] = \left(\frac{32 \times 48}{25}\right)\left(1 - \frac{1}{16}\right) = \frac{32 \times 48}{25} \times \frac{15}{16} = \frac{288}{5}.$$

As a decimal, the area of $CDGE$ is 57.6, or 58 when rounded to the nearest integer.

18. Alice and Brenda each arrive at a diner at a randomly chosen time between 2:00 pm and 3:00 pm, independently of each other. While Alice is patient and willing to wait until Brenda arrives, Brenda will only wait for up to 15 minutes for Alice to arrive, before leaving. The probability that Alice and Brenda meet each other at the diner equals $\dfrac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

    Answer (55): We use geometric probability. Consider choosing a point at random from the square shown below. Then Alice and Brenda will meet each other as long as Alice arrives no later than 15 minutes ($\dfrac{1}{4}$ hour) after Brenda, as shown by the shaded region:

    The desired probability equals the ratio of the shaded area to the area of the square. Considering the region to be a square with side length 1, the area of the unshaded isosceles triangle is $\frac{1}{2} \times \frac{3}{4} \times \frac{3}{4} = \frac{9}{32}$. The area of the shaded region is $1 - \frac{9}{32} = \frac{23}{32}$. The probability is therefore $\frac{23}{32}$, and $m + n = 23 + 32 = 55$.

19. There are four positive integers less than 100 whose squares end in the digits "16." One such integer is 4, as $4^2 = 16$. What are the last two digits of the sum of the other three integers?

    Answer (96): Let x be such an integer, so that $x^2 \equiv 16 \pmod{100}$. The main observation is that x satisfies the given condition if and only if $100 - x$ satisfies the given condition; this follows as $(100 - x)^2 \equiv (-x)^2 \equiv x^2 \pmod{100}$. Therefore, $100 - 4 = 96$ is one such integer; as a check, $96^2 = 9216$. It is not necessary to determine the other two integers since we know that they must sum to 100. Therefore, the sum of the remaining three integers is $100 + 96 = 196$, and the last two digits are 96. We remark that the remaining two integers are 46 and 54.

20. Ann, Beth, and Cia split a large basket containing 100 apples, 100 bananas, and 100 oranges so that each person received 100 pieces of fruit. Ann received one more apple than Beth, and Beth received two more bananas than Cia. Given that Cia received 7 apples, what is the **greatest** whole number of oranges Cia could have received?

    Answer (70): It is easiest to organize our work into a 3×3 table. We are given that Cia received 7 apples and that Ann received one more apple than Beth. Therefore, if Ann received a apples, then Beth received $a - 1$ apples, and $a + (a - 1) + 7 = 100 \implies a = 47$, so Ann and Beth received 47 and 46 apples, respectively:

	Ann	Beth	Cia	
apples	47	46	7	100
bananas				100
oranges				100
	100	100	100	

    Suppose Beth received b bananas. From the problem statement, Cia received $b - 2$ bananas. Since there are 100 bananas altogether, Ann received $100 - b - (b - 2) = 102 - 2b$ bananas. Our table is as follows:

	Ann	Beth	Cia	
apples	47	46	7	100
bananas	$102 - 2b$	b	$b - 2$	100
oranges				100
	100	100	100	

Finally, we can determine the remaining entries in terms of b, since each person received 100 pieces of fruit. The completed table is shown below:

	Ann	Beth	Cia	
apples	47	46	7	100
bananas	$102 - 2b$	b	$b - 2$	100
oranges	$2b - 49$	$54 - b$	$95 - b$	100
	100	100	100	

Cia received $95 - b$ oranges. We want $95 - b$ to be maximized, so we should find the smallest possible integer value of b. Every expression in the table must be a non-negative integer; in particular, $2b - 49 \geq 0 \implies b \geq 25$ (the remaining expressions either yield upper bounds for b, or weaker lower bounds such as $b \geq 2$ or $b \geq 0$). We see that $b = 25$ is indeed a candidate value for b, and the greatest number of oranges Cia received is $95 - 25 = 70$.

21. Let a, b, and c be integers such that

$$\begin{aligned} a + b + c &= 1, \\ 4a + 2b + c &= 8, \text{ and} \\ 9a + 3b + c &= 27. \end{aligned}$$

What is the value of $16a + 4b + c$?

Answer (58): Consider the quadratic polynomial $P(x)$, given by $P(x) = ax^2 + bx + c$. We are given $P(1) = 1$, $P(2) = 8$, and $P(3) = 27$. Then $P(x) = x^3$ has solutions $x = 1$, $x = 2$, and $x = 3$, or in other words, the cubic polynomial $x^3 - P(x)$ has roots $1, 2,$ and 3. Since $x^3 - P(x)$ has leading coefficient 1, it can be factored into the form $(x-1)(x-2)(x-3)$.

To find $P(4)$, we can simply substitute $x = 4$: $64 - P(4) = (4-1)(4-2)(4-3) = 6$. Solving for $P(4)$, we obtain $P(4) = 58$. As a remark, the solution to this system of equations is $(a, b, c) = (6, -11, 6)$.

22. Hannah rolled six standard 12-sided dice (with faces numbered 1 through 12). The product of her dice rolls is 178,200, and no two of Hannah's rolls showed the same number. What is the sum of Hannah's dice rolls?

    Answer (50): First, we will find the prime factorization of 178,200, which is $2^3 \times 3^4 \times 5^2 \times 11^1$. Since no two of Hannah's rolls showed the same number, we will find a way to express 178,200 as the product of six different integers between 1 and 12, inclusive.

    Since the dice rolls are between 1 and 12, and 11 is the only number divisible by the prime number 11, it follows that one of the dice rolls must be 11. Next, observe that the product is divisible by 5^2. Since 5 and 10 are the only two multiples of 5, two of the dice rolls must have been 5 and 10. Therefore, three of the dice rolls are 5, 10, and 11, and the product of the remaining three dice rolls is $2^2 \times 3^4$.

    There are four possible dice rolls which are multiples of 3 (3, 6, 9, and 12); we claim that the remaining three dice rolls must all be multiples of 3. Suppose otherwise at least one die roll is not divisible by 3; in this case, there is no way for the product of the remaining three dice rolls to be divisible by 3^4 (as 9 is the only multiple of 9 between 1 and 12). The product of the numbers 3, 6, 9, and 12 is $2^3 \times 3^5$; it follows that 6 is the multiple of 3 which is not rolled, and that the remaining three dice rolls are 3, 9, and 12.

    The sum of all of Hannah's dice rolls is $5 + 10 + 11 + 3 + 9 + 12 = 50$.

23. 100 people sit at a round table, and every person is either a knight (who always tells the truth), or a knave (who always lies). When every person at the table was asked, "Of the two people sitting next to you, how many of them are knights?", fifty people answered "0," and fifty people answered "2." What is the greatest possible number of knaves who are sitting at the round table?

    Answer (83): Note that if two knights sit next to each other, then every person must be a knight (since no knight can sit next to both a knight and a knave, as this knight would have answered "1"). For this case, every person truthfully answers "2," which contradicts the statement that fifty people answered "0" and fifty answered "2." Thus, we will consider the case when no two knights sit next to each other. Let k be the number of knights. Since we are considering the case where no two knights are adjacent, we have $k \leq 50$.

These k knights truthfully answered "0" as they cannot have answered "2," so $50-k$ knaves must have falsely answered "0." Each of these $50-k$ knaves must be adjacent to at least one knight; otherwise, a knave is adjacent to two knaves and truthfully answers "0" which is a contradiction. Since there are k non-adjacent knights, there are at most $2k$ people who are sitting next to a knight. Since every person sitting next to a knight is a knave, it follows that $50-k \leq 2k$. Solving this inequality for k yields $k \geq \frac{50}{3} = 16.\overline{6}$. Then there are at least 17 knights, and at most $100-17=83$ knaves.

Such an arrangement is possible by seating 17 knave-knight-knave groups in the round table arbitrarily, followed by an additional 49 knaves. The 17 knights, as well as 33 out of the 34 knaves who are sitting next to one knight, answer "0", and the remaining knave, as well as the 49 knaves who are not sitting next to a knight, answer "2."

24. How many non-empty subsets S of the set $\{2,3,5,7,11,13,17,19\}$ are there such that the sum of the elements of S is divisible by 4? One example is $S=\{2,7,11\}$.

Answer (63): Consider choosing a subset T of $\{5,7,11,13,17,19\}$. No matter what subset is chosen, we can uniquely determine a subset S such that $S \subseteq T \cup \{2,3\}$, $T \subseteq S$, and the sum of the elements of S is divisible by 4. To show this, let $s(T)$ denote the sum of the elements of T. If $s(T) \equiv 0 \pmod 4$, then $S=T$. If $s(T) \equiv 1 \pmod 4$, then $S = T \cup \{3\}$. If $s(T) \equiv 2 \pmod 4$, then $S = T \cup \{2\}$. Finally, if $s(T) \equiv 3 \pmod 4$, then $S = T \cup \{2,3\}$. Therefore, by choosing a subset of $\{5,7,11,13,17,19\}$, we can uniquely determine a subset S whose sum is divisible by 4.

Since $|T|=6$, there are $2^6=64$ subsets T, and therefore, $2^6=64$ subsets S whose sum is divisible by 4. However, this counts the empty set $S=\emptyset$, which is selected when $T=\emptyset$. We subtract 1 from our count, giving $64-1=63$ non-empty subsets.

25. A function $f(x)$ satisfies the following properties for all positive integers x:

$$f(x)^2 = f(x^2)+2, \text{ and}$$
$$f(x) = 2f(2x)-3x.$$

The sum of all possible values of $f(8)$ equals $\dfrac{m}{n}$, where m and n are relatively prime positive integers. What are the last two digits of $m+n$?

Answer (73): Substituting $x = 1$ into the first property, we obtain $f(1)^2 = f(1) + 2$, or $f(1)^2 - f(1) - 2 = 0$. This is a quadratic in the variable $f(1)$ which factors as $(f(1)-2)(f(1)+1) = 0$, so either $f(1) = 2$ or $f(1) = -1$.

In both cases, we can compute $f(2)$, $f(4)$, and $f(8)$ successively using the second property, which rearranges to $f(2x) = \frac{f(x)+3x}{2}$. If $f(1) = 2$, we obtain $f(2) = \frac{2+3}{2} = \frac{5}{2}$, $f(4) = \frac{5/2+6}{2} = \frac{17}{4}$, and $f(8) = \frac{17/4+12}{2} = \frac{65}{8}$. We can observe that $f(x) = x + \frac{1}{x}$ in these cases, and the function $f(x) = x + \frac{1}{x}$ satisfies both properties for all positive integers x.

If $f(1) = -1$, then the second property implies $f(2) = \frac{-1+3}{2} = 1$, $f(4) = \frac{1+6}{2} = \frac{7}{2}$, and $f(8) = \frac{7/2+12}{2} = \frac{31}{4}$. However, $f(1) = -1$ creates a contradiction: since $f(2) = 1$, substituting $x = 2$ into the first property implies $f(4) = f(2)^2 - 2 = 1^2 - 2 = -1$. Since $f(4)$ cannot simultaneously equal -1 and $\frac{7}{2}$, this is a contradiction, so $f(1)$ cannot equal -1.

Therefore, the only possible value of $f(8)$ is $\frac{65}{8}$, and $m + n = 65 + 8 = 73$. The last two digits are 73.