Pi Math Contest Euler Division

2024 Solutions

The problems and solutions in this contest were proposed by:

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Solutions

1. Evaluate $6 \div 3 \times 2 - 2$.

   Answer (2): Following the order of operations,

$$\begin{aligned} 6 \div 3 \times 2 - 2 &= (6 \div 3) \times 2 - 2 \\ &= 2 \times 2 - 2 \\ &= 4 - 2 \\ &= 2. \end{aligned}$$

2. Simplify $\dfrac{2024 + 2024 + 2024 + 2024}{2024 + 2024}$.

   Answer (2): We add four occurrences of 2024 in the numerator and two occurrences in the denominator. Then,

$$\begin{aligned} \frac{2024 + 2024 + 2024 + 2024}{2024 + 2024} &= \frac{4 \times 2024}{2 \times 2024} \\ &= \frac{4}{2} \\ &= 2. \end{aligned}$$

3. When Amy's favorite whole number is increased by 2 and then squared, the result is 81. What is her favorite number?

   Answer (7): Recognizing that 9^2 equals 81, we can deduce that Amy's favorite number is two less than 9, which is 7.

4. Tom has a sequence of numbers starting with -5 and increasing by 2 each time: $-5, -3, -1, \ldots$. If the total sum of these numbers is 16, what is the final number in the sequence?

   Answer (9): Tom's sequence, beginning with $-5, -3, -1, 1, 3, 5, \ldots$, pairs negative numbers with their positive counterparts, which cancel each other out. Hence, the sum of the first six numbers $(-5, -3, -1, 1, 3, 5)$ is 0. Continuing this pattern, the next numbers are 7 and 9, whose sum is 16. This indicates the sequence's cumulative total after canceling the preceding pairs. Thus, the last number in Tom's sequence is 9.

5. Lonzo plans to purchase 12 bottles of water and 12 bottles of soda. The water is sold in packs of 4, while the soda is available in packs of 6. What is the total number of packs he needs to buy?

   Answer (5): To find the total packs Lonzo needs, we divide each item by its pack size: 12 bottles of water divided by 4 per pack equals 3 packs, and 12 bottles of soda divided by 6 per pack equals 2 packs. Adding these together, Lonzo needs to buy 5 packs in total.

6. Lily adds $5\frac{2}{5}$ gallons of water to her empty fish tank, filling up three fifths of the tank. What is the total capacity of her fish tank, in gallons?

   Answer (9): $5\frac{5}{2}$ gallons, or $\frac{27}{5}$ gallons, fills three-fifths of the tank. Thus, one-fifth of the tank holds
   $$\frac{27}{5} \div 3 = \frac{9}{5} \text{ gallons.}$$
   Multiplying $\frac{9}{5}$ by 5 gives the tank's total capacity, which is 9 gallons.

7. Rita squares a positive two-digit number, resulting in a three-digit number that is divisible by 3. How many different two-digit numbers could she have started with?

   Answer (7): The square of the two-digit number N should be between 100 and 999, meaning that N is between 10 and 31, inclusive. Since N^2 is divisible by 3,

N must also be divisible by 3. Thus, there are 7 numbers (12, 15, 18, 21, 24, 27, 30) that meet these criteria.

8. What percent of a 24-hour day did Tiger spend on homework if he started at 7:53 PM and finished at 9:05 PM?

   Answer (5): There are 1 hours and 12 minutes (or 72 minutes) from 7:53 PM to 9:05 PM. Since the day has 24×60 minutes, the fraction of the time spent on homework is:
   $$\frac{72}{24 \times 60} = \frac{3}{60} = \frac{1}{20},$$
   which is 5 percent.

9. For how many whole numbers n is the fraction $\frac{n}{12}$ between the fractions $\frac{1}{6}$ and $\frac{5}{8}$?

   Answer (5): Multiplying all fractions by 12, we find that n must be between $\frac{12}{6} = 2$ and $\frac{12\times5}{8} = \frac{60}{8} = 7\frac{4}{8}$. There are 5 numbers satisfying this (3, 4, 5, 6, and 7).

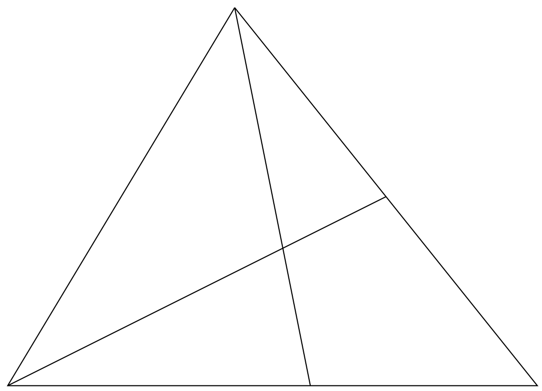
10. What is the units digit of the product $2022 \times 2023 \times 2024$?

    Answer (4): Since we only want the units digit, it is sufficient to find the units digit of the product $2 \times 3 \times 4$, which is 4. We conclude that the units digit of the original product is also 4.

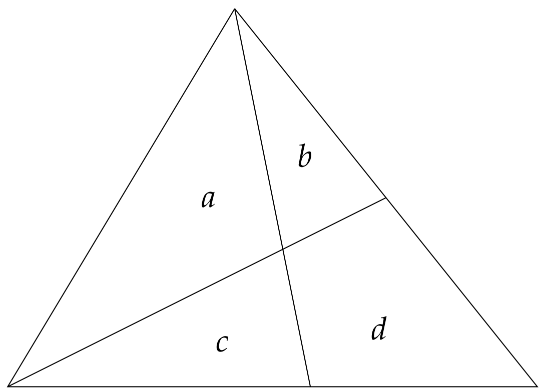
11. The movie "Adventure Swamp" originally cost \$8 but its price increased by 50%. If Helen uses a coupon for 50% off, how many dollars will she pay for the movie?

    Answer (6): After a 50% price increase, the price of the ticket becomes
    $$8 \times 1.5 = 12 \text{ dollars.}$$
    Subsequently, applying a 50% discount coupon results in a final payment of
    $$12 \times 0.5 = 6 \text{ dollars.}$$

12. How many triangles of any size are in the figure below?



Answer (8): Let's label the regions as follows:



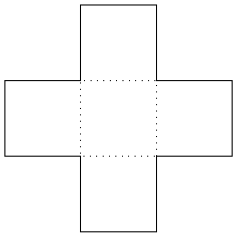
Using these labels, we find 8 triangles in the figure:

$$a,\ b,\ c,\ ac,\ bd,\ ab,\ cd, abcd.$$

13. A class has 10 students, including Freya, Bennett, and Ashley. Freya gives 5 of the students one apple each. Then Bennett gives 5 of the students one orange each. Everyone received at least one fruit except Ashley, who did not receive any fruit. How many students received both an apple and an orange?

    Answer (1): Given that Ashley is the only student who did not receive any fruits, it implies that the remaining 9 students each received at least one fruit. Considering a total of 10 fruits distributed, it follows that only one student received both fruits, as $10 - 9 = 1$.

14. A country has a cross-shaped flag consisting of 5 identical squares, as shown below. If the area of the flag is 1.25 square feet, what is its perimeter, in feet?



Answer (6): The area of each of the five small squares is $\frac{1.25}{5} = \frac{1}{4}$ square feet.

Therefore, the side length of these squares is $\frac{1}{2}$ feet. There are 12 sides in the flag, giving a perimeter of $12 \times \frac{1}{2} = 6$ feet.

15. Forty stickers are distributed among a group of students. Each student receives at least one sticker, and no two students receive the same number of stickers. What is the largest possible number of students in the group?

Answer (8): The student who receives the least number of stickers must receive at least 1 sticker. The student who receives the second least number of stickers must receive at least 2 stickers (since she/he must have at least 1 more sticker than the previous student), and so on. If there are 9 or more students in the group, they would have to distribute at least

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$$

stickers which is more than the total number of stickers. However, if there are 8 students, we can distribute the stickers, for example, as

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 12.$$

Therefore, the largest possible number of students is 8.

16. Alice and Bob, two frogs, alternate turns hopping. Alice hops once on her first turn. On his turn, Bob hops three times the number Alice hopped on her last turn. On her turn, Alice hops twice the number Bob hopped on his last turn. They continue until they have collectively hopped at least 100 times. How many turns did Alice take?

Answer (3): Let's list the number of hops that Alice and Bob take on each respective turn:

frog	hops
Alice	1
Bob	3
Alice	6
Bob	18
Alice	36
Bob	108

Observe that the sum $1 + 3 + 6 + 18 + 36$ equals 64, which is less than 100. However, when we include the next term, 108, the sum becomes greater than 100. Consequently, Alice completes a total of 3 turns.

17. A rectangle has a length of 16 units. An equilateral triangle has a side length of 16 units as well. Given that both the rectangle and the triangle have the same perimeter, what is the width of the rectangle, measured in units?

    Answer (8): The perimeter of the equilateral triangle is $16 \times 3 = 48$ units. Since the rectangle shares the same perimeter, the sum of its length and width must be $48 \div 2 = 24$ units. Therefore, the width of the rectangle is $24 - 16 = 8$ units.

18. LeBron, James, Carmelo, Anthony, and Kyrie participated in a race. All of the following statements are true:

    - James finished in 7^{th} place.
    - LeBron finished two places ahead of James.
    - Kyrie and Lebron finished four places apart.
    - Carmelo and Anthony finished 2 places apart.
    - Kyrie finished seven places behind Anthony.

    What place did Carmelo finish in?

    Answer (4): LeBron finished two places ahead of James, who finished in 7^{th} place. Therefore, LeBron finished in 5^{th} place.

    Since Kyrie and Lebron finished four places apart, either Kyrie finished four places behind LeBron, placing Kyrie in 9^{th} position or finished four places ahead of LeBron, placing Kyrie in 1^{st} position. However, Kyrie finished seven places behind Anthony so he could not be the first. Then, he was 9th.

    Kyrie also finished seven places behind Anthony. Consequently, Anthony secured the 2^{nd} place.

    Carmelo and Anthony finished two places apart. Since there is only one place in front of Anthony, Carmelo could not have finished before Anthony. Thus, Carmelo finished two places behind Anthony, landing in 4^{th} place.

19. Teacher Anders has between 20 and 40 students in his class. When he divides the students in groups of three, 2 students are left over. Similarly, when he divides the students in groups of five, 2 students are left over. How many students will be left over if he divides the class into groups of eight?

    Answer (0): Consider that subtracting the 2 leftover students from the total class count should yield a number that is a multiple of both 3 and 5, essentially a multiple of 15. Within the range of 20 to 40, the only multiple of 15 is 30. Therefore, the class has 32 students, including the 2 leftover students. Dividing 32 students into groups of 8 results in 4 complete groups without any students remaining. Thus, the answer is 0; no students are left over.

20. The product of all positive factors of 128 is equal to 128 raised to which power?

    Answer (4): The factors of 128 can be organized into pairs, each consisting of two numbers whose product equals 128. These pairs are as follows:

    $$1 \times 128,\ 2 \times 64,\ 4 \times 32,\ 8 \times 16.$$

    Each factor pair contributes to the overall product of 128. To find the total product of all factors, we multiply these pairs together, resulting in 128^4. Therefore, the answer is 4.

21. A *geometric* sequence is a sequence where every term is equal to the previous term multiplied by a constant, called the *common ratio*. For example, the sequence 3, 6, 12, 24, . . . is a geometric sequence whose common ratio is 2.

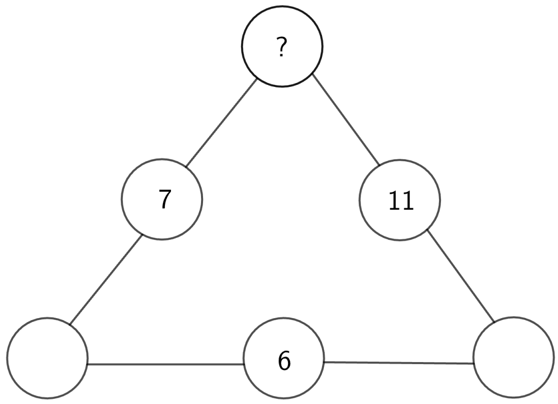
    A geometric sequence with 20 terms has a common ratio of $\frac{2}{5}$. If the 10^{th} term is 200, how many terms of the sequence are whole numbers?

    Answer (6): To analyze the given geometric sequence, let's examine its terms both by moving to the right and by moving to the left.

    Moving to the right, we find that the 11^{th} term is $200 \times \frac{2}{5} = 80$, and the 12^{th} term is $80 \times \frac{2}{5} = 32$. Beyond this point, there are no further factors of 5, resulting in no more integer terms in the sequence.

    On the left side, moving backward, we observe that the 9^{th} term is $200 \div \frac{2}{5} = 500$, the 8^{th} term is $500 \div \frac{2}{5} = 1250$, and the 7^{th} term is $1250 \div \frac{2}{5} = 3125$. Beyond this point, there are no additional factors of 2, leading to the absence of more integer terms. Therefore, including the 10^{th} term, there are a total of 6 whole number terms in the geometric sequence.

22. The diagram below contains six circles, and each circle is to contain one number. The sum of the three numbers on each side of the triangle is 30. What number must occupy the top circle?



Answer (9): Let's consider the sum of the three numbers on each side, fixed at 30. The sum of the two missing numbers on the bottom edge is obtained by subtracting 6 from 30, giving us 24.

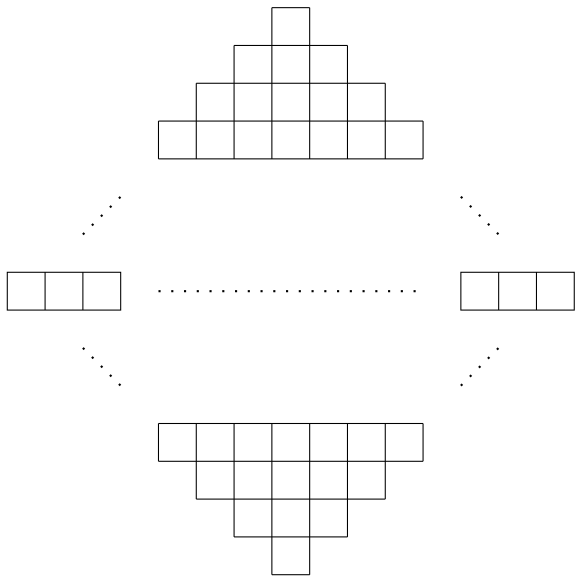
Since 7 and 11, the numbers on the two adjacent sides, differ by 4, we can deduce that the two missing numbers on the bottom edge should also differ by 4. To find these numbers, we look for a pair that adds up to 24 and has a difference of 4. After some consideration, we find that 14 and 10 meet these criteria.

Therefore, we can assign 14 to the bottom left number and 10 to the bottom right number. Now, let's find the top number. Subtracting the known values from the fixed sum of 30, we get the top number as follows:

$$30 - 7 - 14 = 9 \quad \text{or} \quad 30 - 11 - 10 = 9.$$

So, the missing top number is 9.

23. The following figure contains 25 rows: one unit square in the top and bottom rows, 3 unit squares in the second and second to last rows, and so on. If there are N unit squares in total, what is the sum of the digits of N?



Answer (7): Notice that in the upper half of the grid, the number of squares in each row increases by two as you move down, while in the lower half, it decreases by two. Consequently, the middle row, which is the 13th row, contains $1 + 2 \times 12 = 25$ squares.

Therefore, the total number of squares can be expressed as the sum:

$$1 + 3 + 5 + \cdots + 23 + 25 + 23 + \cdots + 5 + 3 + 1.$$

This sum can be rearranged as:

$$2 \times (1 + 3 + 5 + \cdots + 21 + 23) + 25.$$

By pairing smaller and bigger numbers in the sum, we get:

$$\begin{aligned} N &= 2 \times ((1 + 23) + (3 + 21) + \cdots + (11 + 13)) + 25 \\ &= 2 \times 24 \times 6 + 25 \\ &= 313. \end{aligned}$$

The sum of the digits of 313 is $3 + 1 + 3 = 7$.

24. Vera constructs a sequence as follows: she initiates the sequence with two positive integers, not necessarily distinct. Every term thereafter is obtained by adding the two immediately preceding terms. For example, if the initial pair of numbers were 1 and 3, the resulting sequence would be 1, 3, 4, 7, 11, and so on. It is given that the fifth term in Vera's sequence is 30. Determine the

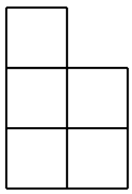
total number of distinct sequences Vera could have generated under these conditions.

Answer (4): Let the first two terms of the sequence be denoted as a and b. Then the third term is $a+b$, the fourth term is $b+(a+b)=a+2b$, and the fifth term is $(a+b)+(a+2b)=2a+3b$. Moreover, the fifth term is 30, so we have the equation $2a+3b=30$. We can rewrite this equation as follows:

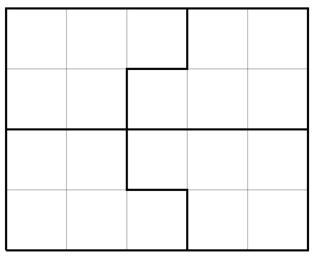
$$\begin{aligned} 2a+3b &= 30 \\ 2a &= 30-3b \\ a &= \frac{3(10-b)}{2} \end{aligned}$$

To have a positive integer for a, the number $(10-b)$ must be even and positive. Therefore, b can be $2,4,6,8$. For each value of b, we can form a unique sequence, so there are 4 such sequences.

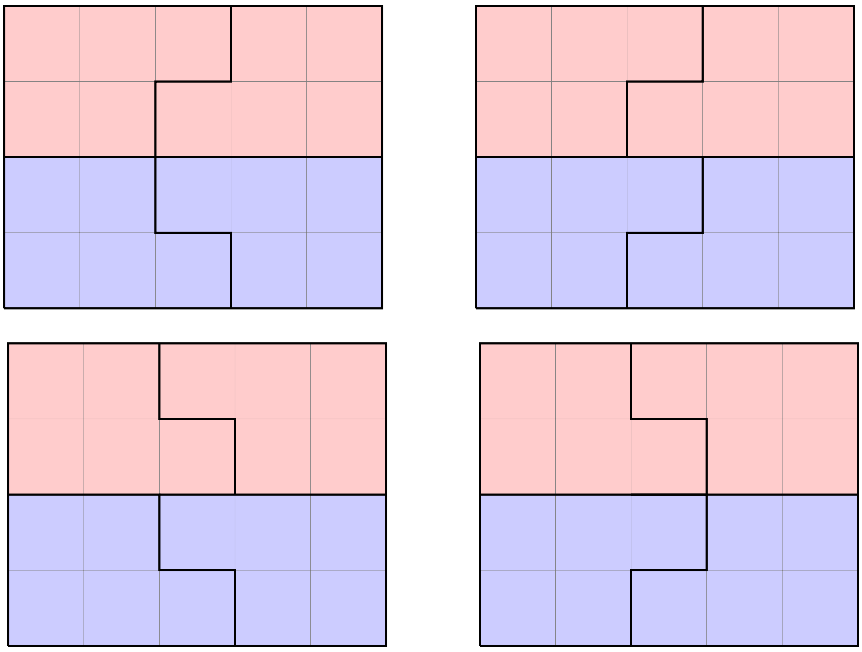
25. Consider the following pentomino created by adjoining a unit square to a 2×2 square:



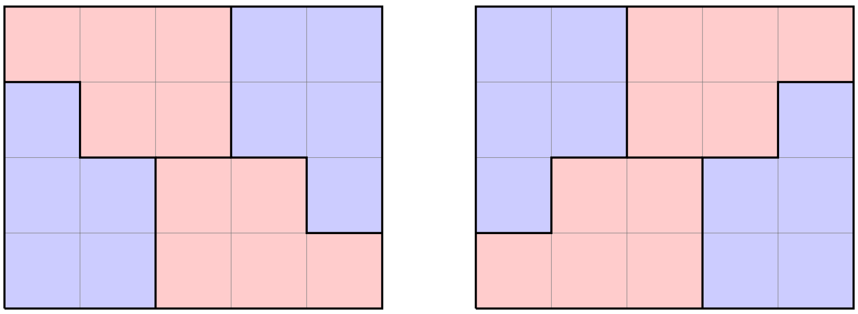
How many ways are there to tile a 5×4 grid with four of these pentominoes without overlap, including the example below? (The pentominoes may be rotated or flipped.)



Answer (6): Through a bit of exploration, we can uncover two distinct types of tilings. The initial type involves tiling the two 5×2 grids within the larger 5×4 grid. In this scenario, there exist four possible arrangements, as each 5×2 grid can be oriented in two different ways, resulting in a total of $2\cdot 2=4$ tiling variations:



The second kind involves putting two pentominoes flush to opposite corners the other way, and the rest becomes apparent; there are 2 ways here:



The total number of ways is 6.