Pi Math Contest Gauss Division

2023

Solutions

1. Evaluate
$$111 - 222 + 333 - 444 + 555 - 666 + 777 - 888 + 999.$$

Answer (555): Pairing the numbers in the following way, we get

$$\begin{aligned}
& 111 - 222 + 333 - 444 + 555 - 666 + 777 - 888 + 999 \\
= \; & 111 + (-222 + 333) + (-444 + 555) + (-666 + 777) + (-888 + 999) \\
= \; & 111 + 111 + 111 + 111 + 111 \\
= \; & 555.
\end{aligned}$$

2. What is the value of
$$24 + \frac{120}{6 + \frac{1}{1+\frac{1}{2}}}?$$

Answer (42): Working our way from the bottom to the top, we obtain:

$$\begin{aligned}
24+\cfrac{120}{6+\cfrac{1}{\mathbf{1}+\cfrac{\mathbf{1}}{\mathbf{2}}}} &= 24+\cfrac{120}{6+\cfrac{\mathbf{1}}{\mathbf{3/2}}} \\
&= 24+\cfrac{120}{\mathbf{6}+\cfrac{\mathbf{2}}{\mathbf{3}}} \\
&= 24+\frac{\mathbf{120}}{\mathbf{20/3}} \\
&= 24+\mathbf{120}\cdot\frac{\mathbf{3}}{\mathbf{20}} \\
&= 24+18 \\
&= 42.
\end{aligned}$$

3. There are 15 blue, 35 red, 47 green, and 37 purple balls in a bag. Balls are picked one at a time without replacement. What is the fewest number of balls that need to be picked in order to ensure that balls of at least three different colors have been picked?

   Answer (85): Note that the answer is 1 more than the largest number of balls that can be picked using only two colors. Since green and purple have the largest number of balls, the answer is $47 + 37 + 1 = 85$.

4. The sum of four different positive integers is 154. What is the least possible value of the largest integer?

   Answer (40): Let the largest number be a. Since the numbers are distinct, the sum can be at most

   $$a + (a - 1) + (a - 2) + (a - 3) = 4a - 6.$$

   So, we must have $4a - 6 \geq 154$ which implies $a \geq 40$.

   For $a = 40$, the numbers 37, 38, 39, and 40 work. So, the answer is 40.

5. Doctor King and nurse Light are emergency room personnel. Doctor King has a night duty every 8 days and nurse Light has a night duty every 6 days. If they have their first night duty on Monday together, in how many days will they have their next Monday night duty together?

Answer (168): The number of days between two consecutive common night shifts is the least common multiple of 8 and 6, which is 24. So, they have night shifts together every 24 days.

Since we are looking for the next **Monday** night duty together, the number of days passed must also be a multiple of 7.

Therefore, the answer is the least common multiple of 24 and 7, which is $24 \cdot 7 = 168$.

6. Alice and Bob start on the number line at 10 and 1000, respectively. They walk towards each other. If Alice is walking twice as fast as Bob, where will they meet on the number line?

   Answer (670): Since Alice is walking twice as fast as Bob, she will cover twice as much distance in the same time period. The total distance walked by the two is $1000 - 10 = 990$. So, $\frac{2}{3}$ of this is covered by Alice while $\frac{1}{3}$ of it is covered by Bob. It means that Alice has walked a distance of $\frac{2}{3} \times 990 = 660$, and will be at $10 + 660 = 670$ when they meet.

7. Teacher Abby went to the store to buy supplies for her students. She spent exactly \$999 to buy notebooks and pens. Each pen costs \$5 and each notebook costs \$7. Find the difference between the largest number of items and the smallest number of items that she could have purchased.

   Answer (56): The number of items purchased will be maximized when she buys as many pens as possible since pens cost less. So, we let her buy one notebook at a time until the remaining money is a multiple of \$5. We achieve this with 2 notebooks, leaving \$985 to buy 197 pens. So, the largest number of items she could purchase is $197 + 2 = 199$.

   On the other hand, to minimize the number of items purchased, she should buy one pen at a time until the remaining money is a multiple of \$7. This leads to 1 pen and $\frac{994}{7} = 142$ notebooks, or 143 items.

   Therefore, the answer is $199 - 143 = 56$.

8. There are m ways to arrange the letters of $ALPHASTAR$ and n ways to arrange the letters of $ACADEMY$. Find $\frac{m}{n}$.

   Answer (24): In $\boldsymbol{A}LPH\boldsymbol{A}ST\boldsymbol{A}R$, letter $\boldsymbol{A}$ is repeating 3 times and all the other letters are distinct. By repeated permutation, the number of distinct arrangements is $\frac{9!}{3!}$. This is because if all the letters were different (like $A_1LPHA_2STA_3R$) there would be 9! different arrangements. However, each arrangement here is counted 3!

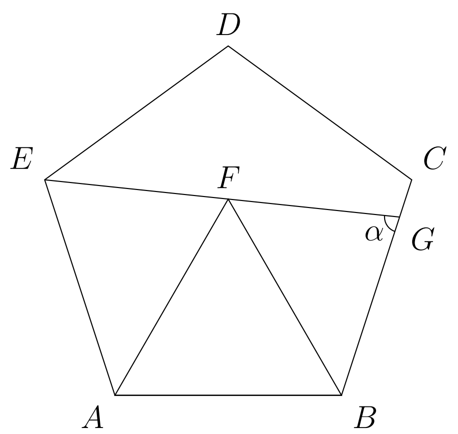
times due to different orderings of A_1, A_2, and A_3. In order to fix this, we divide 9! by 3!.

With the same process, we find that the number of distinct arrangements for $\boldsymbol{A}C\boldsymbol{A}DEMY$ is $\frac{7!}{2!}$.

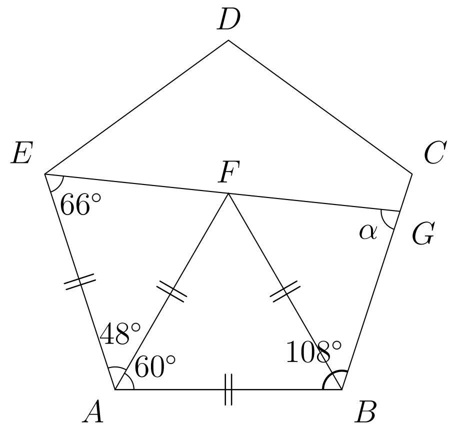
We see that $m = \frac{9!}{3!}$ and $n = \frac{7!}{2!}$ and we simplify $\frac{m}{n}$ as follows:

$$\begin{aligned} \frac{m}{n} &= \frac{9!/3!}{7!/2!} \\ &= \frac{9!}{7!} \cdot \frac{2!}{3!} \\ &= \frac{9 \cdot 8}{3} \\ &= 24. \end{aligned}$$

9. In the figure below, $ABCDE$ is a regular pentagon and ABF is an equilateral triangle. Points E, F, G are on the same line. What is the measure of the angle α in degrees?



Answer (78):



The sum of the angles of an n-gon is $180(n-2)^\circ$, hence one angle of the regular pentagon is $\frac{180\times 3}{5} = 108$ degrees. $\triangle AFB$ is equilateral with all of its angles 60°. This implies

$$\angle EAF = 108^\circ - 60^\circ = 48^\circ.$$

Let's look at $\triangle EAF$. It is isosceles with $AE = AF$ because the equilateral triangle and the regular pentagon share a side. Since $\angle EAF = 48^\circ$, we have

$$\angle AEF = \angle AFE = \frac{180^\circ - 48^\circ}{2} = 66^\circ.$$

Lastly, since the angles of the quadrilateral $ABGE$ sum to 360, we get

$$66^\circ + 2 \times 108^\circ + \alpha = 360^\circ,$$

and $\alpha = 78^\circ$.

10. Eight teachers are attending a professional development day. They will work in pairs during the day. In how many ways can they be paired up? The order of the pairs is not important.

    Answer (105): The number of possible pairings can be calculated using the multiplication principle. Let's name the teachers as A, B, C, D, E, F, G, and H. There are 7 choices for who can be paired up with A. Suppose A's teammate is B. Then there are 5 choices for who can be paired up with C. Let's say C is paired up with D. Then there are 3 choices for who can be paired up with E. If E is paired up with F, then there is only one option for the fourth pair: G and H. Using the multiplication principle, there are $7 \cdot 5 \cdot 3 \cdot 1 = 105$ ways for pairing the teachers.

Alternate Solution: We use over-counting technique. Let's line up all the teachers in a row as $ABCDEFGH$ and pair them as 1st-2nd (AB), 3rd-4th (CD), 5th-6th (EF), and 7th-8th (GH). There are 8! ways to order the teachers in a line. But each pairing is counted $2^4 \cdot 4!$ times because the members of each pair can be swapped leading to 2^4 ways and the 4 pairs can be rearranged in 4! ways. This leads to $\frac{8!}{2^4 \times 4!} = 105$ ways for pairing the teachers.

11. Emma is writing an article comprised of four paragraphs of 300 words each. She types the first paragraph at a pace of 60 words per minute. Her pace increases by 20 words per minute on each successive paragraph. To the nearest integer, what is Emma's average pace over the whole article, in words per minute?

    Answer (84): To find the average, we need to divide the total number of words in the three paragraphs by the total time. The total number of words is $300 \times 4 = 1200$. She writes the first paragraph in $\frac{300}{60} = 5$ minutes, the second paragraph in $\frac{300}{80} = 3.75$ minutes, the third paragraph in $\frac{300}{100} = 3$ minutes, and the fourth paragraph in $\frac{300}{120} = 2.5$ minutes. So the total time is 14.25 minutes. Her average pace in words per minute is then

$$\frac{1200}{14.25} = \frac{4800}{57} = \frac{1600}{19} \simeq 84.2 \simeq 84.$$

    Alternate Solution: We can find the average pace using harmonic mean since all the paragraphs have the same number of words. Her paces are 60, 80, 100 and 120 words per minute. Therefore, her average pace (in words per minute) is

$$\frac{4}{\frac{1}{60} + \frac{1}{80} + \frac{1}{100} + \frac{1}{120}} = \frac{4 \cdot 1200}{57} \simeq 84.$$

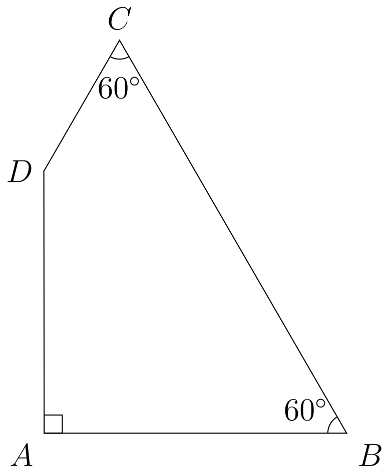
12. A twin prime is a prime number that is either 2 less or 2 more than another prime number. For example, 3 and 5 are twin primes. What is the sum of the two largest two-digit twin primes?

    Answer (144): We check for prime numbers starting with 99 and going down until we find two that differ by 2. The list is

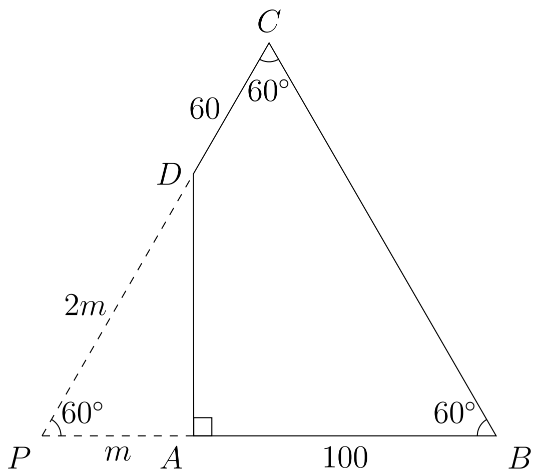
$$97,\ 89,\ 83,\ 79,\ \mathbf{73},\ \mathbf{71}.$$

    We find that the largest twin primes are 73 and 71. Their sum is 144.

13. In the figure below, $ABCD$ is a quadrilateral with $\angle A = 90°$ and $\angle B = \angle C = 60°$. If $AB = 100$ and $CD = 60$, what is BC?



Answer (140):



We extend the lines $\overline{CD}$ and $\overline{BA}$ to intersect at P. Note that $\triangle CPB$ is equilateral and $\triangle DPA$ is a 30-60-90 special triangle. Letting $AP = m$, $DP = 2m$, and using $BP = CP$ we obtain

$$m + 100 = 2m + 60.$$

Solving this we get $m = 40$. The answer is

$$BC = BP = m + 100 = 140.$$

14. Let $f(x) = x + 1 + \sqrt{4x}$ for non-negative real numbers x. Determine the value of a for which $f(f(f(a))) = 1024$.

Answer (841): We can rewrite $f(x)$ as

$$f(x) = x + 1 + 2\sqrt{x} = (\sqrt{x} + 1)^2.$$

From this, we note that $f(k^2) = (k+1)^2$, so if x is a perfect square, then $f(x)$ is the next square. Therefore,

$$f(f(f(a))) = 1024 = 32^2$$

means that $1024 = 32^2$ is the next next next square after a, or a is the third square before 32^2. We deduce that

$$a = 29^2 = 841.$$

15. Let $a \triangleright b = a^2b - 3a^2 - 9b + 37$ for real numbers a and b. Find

$$1 \triangleright (2 \triangleright (3 \triangleright (\cdots \triangleright (2022 \triangleright 2023) \cdots))).$$

Answer (234): We factorize $a^2b - 3a^2 - 9b + 37$ as follows:

$$\begin{aligned} a \triangleright b &= a^2b - 3a^2 - 9b + 37 \\ &= (a^2b - 3a^2) + (-9b + 27) + 10 \\ &= a^2(b-3) - 9(b-3) + 10 \\ &= (b-3)(a^2-9) + 10 \end{aligned}$$

Observe that if $a = 3$ then $a \triangleright b = 10$ for any real number b. Therefore, it remains to compute:

$$1 \triangleright (2 \triangleright 10) = 1 \triangleright (-25) = 234.$$

16. The number 3 is written on a blackboard. At each step, every number n on the board is replaced with the numbers $n+1$ and $3n-1$. For example, in the first step, 3 is replaced with 4 and 8. What is the average of all of the numbers on the board after eight steps?

Answer (768): After the first step, the numbers are 4 and 8 with an average of 6. In the second step, 4 is replaced by 5 and 11, and 8 is replaced by 9 and 23.

The average of these four numbers (5, 11, 9, and 23) is 12. Observe that the average is doubled after each step. Since the average of the two numbers that any number n is replaced by is $\frac{(n+1)+(3n-1)}{2} = 2n$, the overall average indeed doubles after every step. Since the average is 3 at the beginning, it will be $3 \cdot 2^8 = 768$ after eight steps.

17. For a positive integer N which is at least 10, two of its decimal digits are chosen randomly. Let p_N be the probability that these digits are identical and q_N be the probability that they are distinct. We call N a *balanced number* if $p_N = q_N$. Find the number of balanced numbers between 10 and 10000, inclusive.

Answer (324): Let k be the number of digits of a balanced number. If a denotes the number of pairs of digits that are the same and b denotes the number of pairs of digits that are different then we have

$$p_N = \frac{a}{\binom{k}{2}} \quad \text{and} \quad q_N = \frac{b}{\binom{k}{2}}.$$

Combining $p_N = q_N$ and $a + b = \binom{k}{2}$, we deduce that

$$a = b = \frac{\binom{k}{2}}{2}.$$

Note that this forces $\binom{k}{2}$ to be even. Both $\binom{2}{2} = 1$ and $\binom{3}{2} = 3$ are odd numbers, thus k must be at least 4.

When $k = 4$, we have $a = b = \frac{\binom{4}{2}}{2} = 3$. The only way for this is for the number to have three identical digits and a fourth different digit. There are $9 \cdot 9$ ways to choose the two distinct digits (first choose the leftmost digit, then choose the other digit), and there are 4 ways to choose which digit place will have the unique digit. This gives $4 \cdot 9 \cdot 9 = 324$ balanced numbers. Finally, note that 10000 is not a balanced number. Hence, the answer is 324.

18. Let r, s, and t be the roots of the cubic polynomial $x^3 - 15x + 4$. Find $r^4 + s^4 + t^4$.

Answer (450): First, let's look for an integer root. If $x^3 - 15x + 4 = 0$ for an integer x, then $x(x^2 - 15) = 4$, so x is a divisor of 4, possibly negative. Checking the divisors, we see that $x = -4$ works. Thus, $x + 4$ is a factor of the polynomial. Dividing the polynomial by $x + 4$ we get

$$x^3 - 15x + 4 = (x + 4)(x^2 - 4x + 1).$$

Using the quadratic formula, we find that the other two roots are $2 \pm \sqrt{3}$. Now, we find the sum of the fourth powers of the roots:

$$\begin{aligned} & (-4)^4 + (2+\sqrt{3})^4 + (2-\sqrt{3})^4 \\ = \; & 16^2 + (7+4\sqrt{3})^2 + (7-4\sqrt{3})^2 \\ = \; & 256 + (97+56\sqrt{3}) + (97-56\sqrt{3}) \\ = \; & 450. \end{aligned}$$

Alternate Solution: If x is a solution to $x^3 - 15x + 4 = 0$, then multiplying both sides by x and isolating the x^4 term, we obtain

$$x^4 = 15x^2 - 4x.$$

Summing this equation over r, s, and t we get

$$r^4 + s^4 + t^4 = 15(r^2 + s^2 + t^2) - 4(r + s + t).$$

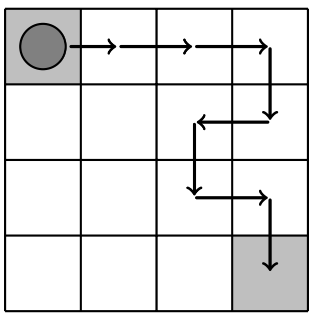
By Vieta's relations, we have $r + s + t = 0$ and $rs + st + tr = -15$. Using these, we can find the sum of squares:

$$r^2 + s^2 + t^2 = (r + s + t)^2 - 2(rs + st + tr) = 0 - 2(-15) = 30.$$

After substituting these values, we deduce that

$$r^4 + s^4 + t^4 = 15 \cdot 30 - 4 \cdot 0 = 450.$$

19. A token is placed in the top left corner of a 4 by 4 checkerboard. A step consists of moving the token one space up, down, to the left, or to the right, so long as it doesn't move off of the board. Find the number of 8-step sequences that move the token to the bottom right square. (Note that a sequence can create a self-intersecting path.)



Answer (336): Note that the token either takes an extra move each to the left and to the right, or an extra move each upwards and downwards. These cases are symmetric, so let us just consider upwards and downwards. Expressing the path in letters, we have 1 U, 4 D, and 3 R to make a path (where each letter corresponds to a direction), provided that U is not before or after all of the D's (in which case the token moves off of the board). Let us first replace the U with a D; then there are $\binom{8}{3}$ ways to create the path. There are 3 ways to relabel one of the D's to a U so that the U is not before or after any D's. In all (multiplying by 2 for the other symmetric case), the number of such paths is

$$2 \cdot 3 \cdot \binom{8}{3} = 336.$$

20. A sequence (a_n) has the property that

$$a_1 + a_2 + \cdots + a_n = n^3 \text{ for all positive integers } n.$$

Let (b_n) be the sequence such that

$$b_1 + b_2 + \cdots + b_n = a_n \text{ for all positive integers } n.$$

Let (c_n) be the sequence such that

$$c_1 + c_2 + \cdots + c_n = b_n \text{ for all positive integers } n.$$

Find c_{2023}.

Answer (6): We will find the first terms of each sequence to see a pattern. Let's start with the sequence (a_n). Subtracting the equations $a_1 + a_2 + \cdots + a_n = n^3$ and $a_1 + a_2 + \cdots + a_{n-1} = (n-1)^3$, we obtain $a_n = n^3 - (n-1)^3$. Using this, we get the following values:

$n:$	1	2	3	4	5	$\cdots$
$a_n:$	1	7	19	37	61	$\cdots$

Next, we find the b_n terms. Similarly, we have $b_n = a_n - a_{n-1}$ for $n \geq 2$. Using this along with $b_1 = a_1 = 1$, we get

$n:$	1	2	3	4	5	$\cdots$
$b_n:$	1	6	12	18	24	$\cdots$

Observe that aside from b_1, the sequence (b_n) is an arithmetic sequence with a common difference of 6. This pattern indeed continues because for $n \geq 1$ we have

$$a_n = n^3 - (n-1)^3 = 3n^2 - 3n + 1$$

and for $n \geq 2$ we have

$$b_n = a_n - a_{n-1} = (3n^2 - 3n + 1) - (3(n-1)^2 - 3(n-1) + 1) = 6n - 6.$$

Finally, we look at (c_n). As above, we have $c_n = b_n - b_{n-1}$ for $n \geq 2$. Using this along with $c_1 = b_1 = 1$, we get

$n:$	1	2	3	4	5	$\cdots$
$c_n:$	1	5	6	6	6	$\cdots$

Observe that after the first two terms, c_n becomes the constant sequence of 6. This is because $c_n = b_n - b_{n-1}$ and after the first term, (b_n) is an arithmetic sequence with a common difference of 6. Therefore, the answer is $c_{2023} = 6$.

21. Let the *scope* of a rectangular prism be the sum of its volume, surface area, and lengths of its 12 edges. What is the sum of integers n, $1 \leq n \leq 60$, for which there exists a rectangular prism with integer side lengths and scope n?

Answer (333): Note that that the *scope* of an $a \times b \times c$ rectangular prism is

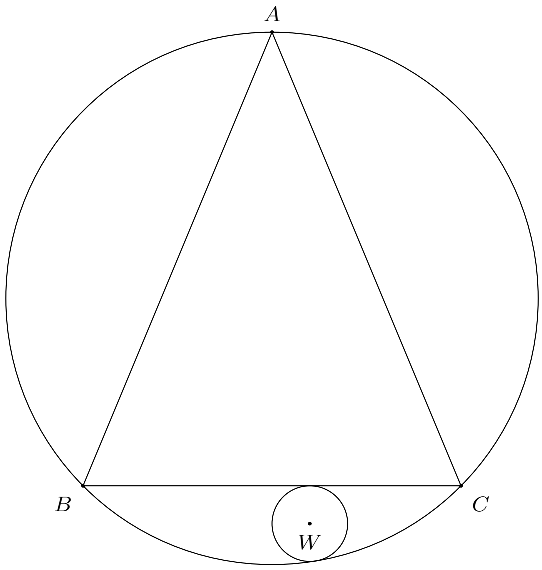
$$abc + 2ab + 2ac + 2bc + 4a + 4b + 4c = (a+2)(b+2)(c+2) - 8.$$

To have a *scope* at most 60, we need $(a+2)(b+2)(c+2) \leq 68$, so we are looking for integers that are at most 68 which are equal to the product of three integers greater than or equal to 3. We list out all possible cases below:

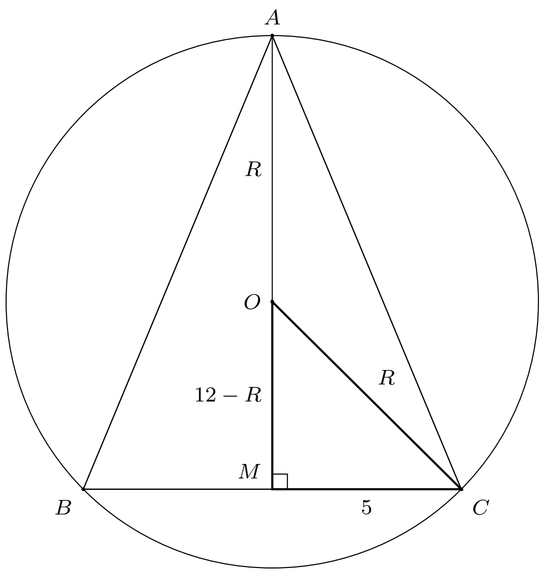
$a+2$	$b+2$	$c+2$	product	*scope*
3	3	3	27	19
3	3	4	36	28
3	3	5	45	37
3	3	6	54	46
3	3	7	63	55
3	4	4	48	40
3	4	5	60	52
4	4	4	64	56

Summing these possible *scopes*, we get 333.

22. In the figure below, $\triangle ABC$ is isosceles with $AB = AC = 13$ and $BC = 10$. Circle ω is a unit circle centered at W that is tangent to $\overline{BC}$ and internally tangent to the circumcircle of $\triangle ABC$. Find AW^2.



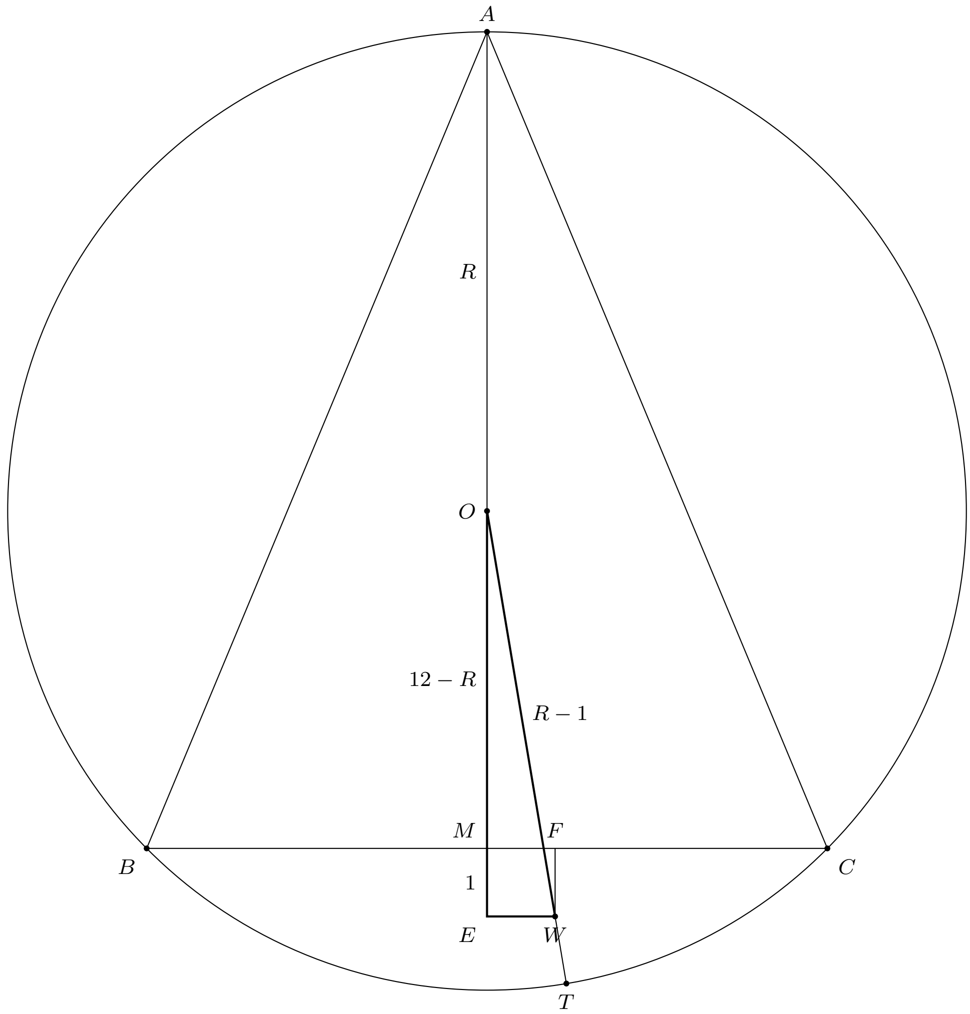
Answer (170):



We will first find R, the circumradius of $\triangle ABC$. Let O be the circumcenter of $\triangle ABC$ and M be the midpoint of $\overline{BC}$. Note that O is on $\overline{AM}$ (the perpendicular bisector) because $AB = AC$. Using Pythagorean theorem on $\triangle OMC$ we get

$$(12 - R)^2 + 5^2 = R^2.$$

Solving this, we obtain $R = \frac{169}{24}$.



Next, let T be the intersection point of ω and the circumcircle of $\triangle ABC$. Points O, W and T are on the same line because the two circles are tangent. Hence

$$OW = OT - WT = R - 1.$$

Let E and F be the feet of perpendiculars from W to $\overline{AM}$ and $\overline{BC}$, respectively. $EWFM$ is a rectangle, hence

$$ME = WF = 1 \text{ and } OE = OM + ME = 13 - R.$$

Using Pythagorean theorem on OEW, we get

$$(13 - R)^2 + EW^2 = (R - 1)^2 \implies EW^2 = (R - 1)^2 - (13 - R)^2 = 24R - 168.$$

Plugging in $R = \frac{169}{24}$ gives $EW = 1$. (Note that this means that E is actually on ω.)

Lastly, using Pythagorean theorem once more, this time on $\triangle AWE$, with $AE = 13$ and $EW = 1$, we obtain

$$AW^2 = 13^2 + 1^2 = 170.$$

Remark: The circumradius of $\triangle ABC$ can also be calculated using the formula $R = \frac{abc}{4[ABC]}$, where a, b, c denote the side lengths of $\triangle ABC$, and $[ABC]$ denotes its area.

23. Twelve chairs are evenly spaced around a circular table. Four distinct chairs are randomly chosen. The probability that no two adjacent chairs are chosen is $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Answer (40): Let's number the chairs from 1 to 12. When we select four chairs (say chairs with numbers a, b, c, and d) so that no two are adjacent, this creates 5 intervals A, B, C, D, and E, where A is the numbers 1 through $a - 1$, B is the numbers $a + 1$ through $b - 1$, and so on, finally E being the numbers $d + 1$ through 12. Note that the non-adjacent condition implies that B, C, and D must all be non-empty, and moreover, A and E cannot both be empty (otherwise, two adjacent numbers 1 and 12 would be selected).

There are a total of $12 - 4 = 8$ numbers on these five intervals. So, this is the same as counting the number of solutions to

$$A + B + C + D + E = 8$$

over non-negative integers such that only A or E can be zero but not both.

If they are all positive integers, by stars and bars, there are $\binom{7}{4} = 35$ solutions. If A or E is 0, then there are $2 \cdot \binom{7}{3} = 70$ solutions. In total, there are 105 solutions.

The number of all possibilities when choosing four distinct chairs is $\binom{12}{4} = 495$. So, the desired probability is $\frac{105}{495} = \frac{7}{33}$, and the answer is $7 + 33 = 40$.

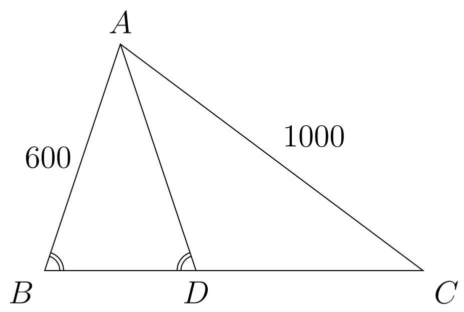
Alternate solution: For simplicity we fix one of the people's positions. Then we can insert the other three people in $\binom{11}{3}$ ways. Now let's find out how many of these lead to non-adjacent choices. Let there be a, b, c, d chairs between the first and second, second and third, third and fourth, and fourth and first people, respectively. We would like a, b, c, d to be positive integers satisfying $a+b+c+d = 8$. By stars and bars, there are $\binom{7}{3}$ ways to select (a, b, c, d), which uniquely defines the positions of the remaining three people.

Thus the probability we desire is

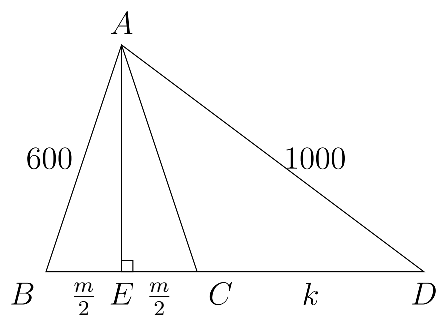
$$\frac{\binom{7}{3}}{\binom{11}{3}} = \frac{35}{165} = \frac{7}{33}.$$

As before, the answer is $7 + 33 = 40$.

24. In the figure below, $AB = 600$, $AC = 1000$, and $\angle ABD = \angle ADB$. The triangles ABC, ABD and ADC have integer perimeters. What is the largest possible value of CD?



Answer (640):



Let $BC = m$ and $CD = k$. Note that the integer perimeters condition is equivalent to m and k being both positive integers. Let E be the foot of the perpendicular from A onto $\overline{BC}$. $\triangle ABC$ is isosceles, hence $BE = EC = \frac{m}{2}$. Pythagorean theorem on $\triangle AEB$ and $\triangle AED$ implies

$$\left(\frac{m}{2}\right)^2 + AE^2 = 600^2$$
$$\left(k + \frac{m}{2}\right)^2 + AE^2 = 1000^2.$$

Subtracting the equations and using the difference of squares yields

$$k(m + k) = 400 \times 1600 = 800^2.$$

On one hand $k^2 < k(m + k) = 800^2$, thus $k < 800$. On the other hand, k is a divisor of 800^2. It suffices to choose k as the largest divisor 800^2 less than 800.

Since k is a divisor of $800^2 = 2^{10}5^4$, we have $k == 2^a5^b$, where $0 \le a \le 10$ and $0 \le b \le 4$. We do casework on b. When $b = 0, 1, 2, 3$ or 4 we find the largest k that is less than 800 as $k = 512$, 640, 400, 500, and 625, respectively. The answer is the largest of these which is 640.

25. The sequence a_1, a_2, a_3, ... satisfies $a_1 = -720$ and

$$a_n = a_{n+1} + a_{n-1} \text{ for } n \ge 2.$$

If $a_8 = 10a_1 + 9a_2 + 8a_3 + \cdots + a_{10}$, what is a_2?

Answer (72): We can rewrite the recurrence relation as $a_n = a_{n-1} - a_{n-2}$ for $n \ge 3$.

For $k \ge 3$, by telescoping we have

$$\begin{aligned} & a_1 + \cdots + a_{k-1} + a_k \\ &= a_1 + a_2 + (a_2 - a_1) + \cdots + (a_{k-2} - a_{k-3}) + (a_{k-1} - a_{k-2}) \\ &= a_2 + a_{k-1}. \end{aligned}$$

Summing this identity over k values, we get

$$\begin{aligned} 10a_1 + 9a_2 + 8a_3 + \cdots + a_{10} &= a_1 + (a_1 + a_2) + \cdots + (a_1 + a_2 + \cdots + a_{10}) \\ &= a_1 + (a_2 + a_1) + (a_2 + a_2) + \cdots + (a_2 + a_8) + (a_2 + a_9) \\ &= a_1 + 9a_2 + (a_1 + a_2 + \cdots + a_9) \\ &= a_1 + 9a_2 + a_2 + a_8 \\ &= a_1 + 10a_2 + a_8. \end{aligned}$$

Since this value is equal to a_8, we must have $a_1 + 10a_2 = 0$, so

$$a_2 = -\frac{a_1}{10} = 72.$$