Pi Math Contest Fermat Division

2023 Solutions

The problems and solutions in this contest were proposed by:

Ali Gurel, Edwin Peng, Edwin Xie, Emre Gul, Isaac Li, Kelly Cui, and Stan Zhang.

Solutions

1. What is the value of

$$23 - 12 + 37 - 18 + 25?$$

Answer (55): To make the calculations faster, we can group the numbers as follows:

$$\begin{aligned} & 23 - 12 + 37 - 18 + 25 \\ = & (23 + 37) - (12 + 18) + 25 \\ = & 55. \end{aligned}$$

2. Simplify

$$\left(\frac{2}{5} + \frac{2}{3}\right) \div \frac{1}{30}.$$

Answer (32): Note that dividing by $\frac{1}{30}$ is the same as multiplying by 30. So, we have

$$\left(\frac{2}{5} + \frac{2}{3}\right) \cdot 30.$$

Expanding this, we get

$$\frac{2}{5}\cdot 30+\frac{2}{3}\cdot 30=2\cdot\frac{30}{5}+2\cdot\frac{30}{3}$$

$$= 2\cdot 6+2\cdot 10=12+20=32.$$

3. What is 68% of 50?

   Answer (34): Note that 68% of 50 is the same as 50% of 68, which is half of 68, or 34.

4. Simplify

$$\frac{2^{20}+2^{23}}{2^{18}}.$$

   Answer (36): Dividing both terms by 2^{18} we get

$$\frac{2^{20}+2^{23}}{2^{18}} = 2^2+2^5=4+32=36.$$

5. Three of the angles of a quadrilateral have degree measures 57°, 79°, and 143°. What is the degree measure of the fourth angle?

   Answer (81): The sum of the angles in a quadrilateral is 360°. The sum of the three given angles is

$$57^\circ+79^\circ+143^\circ=(57^\circ+143^\circ)+79^\circ=279^\circ.$$

   So, the remaining angle must be

$$360^\circ-279^\circ=81^\circ.$$

6. Abby and Betty are eating from a bag of 42 crackers. Abby eats $\frac{1}{3}$ of the crackers and Betty eats $\frac{3}{7}$ of the remaining crackers. How many crackers are left in the bag?

Answer (16): After Abby eats $42 \cdot \frac{1}{3} = 14$ crackers, $42 - 14 = 28$ crackers are left. Betty then eats $\frac{3}{7}$ of these, which is

$$28 \cdot \frac{3}{7} = 3 \cdot \frac{28}{7} = 3 \cdot 4 = 12 \text{ crackers.}$$

In the end, $28 - 12 = 16$ crackers are left.

Alternate Solution: After eating $\frac{1}{3}$ of the crackers, $\frac{2}{3}$ are left. Similarly, after eating $\frac{3}{7}$ of the crackers, $\frac{4}{7}$ are left. So, in the end $\frac{4}{7}$ of $\frac{2}{3}$ of 42 crackers are left, which is

$$\begin{aligned} & \left(42 \cdot \frac{2}{3}\right) \cdot \frac{4}{7} \\ = \; & 28 \cdot \frac{4}{7} \\ = \; & 16. \end{aligned}$$

7. What is the least positive odd integer such that when it is added to 2023, the result is a perfect square?

   Answer (93): Note that $45^2 = 2025$ is a perfect square but it is 2 (an even number) more than 2023. The next perfect square is $46^2 = 2116$, which is 93 more than 2023. Since 93 is odd, it is the answer.

8. For a real number x, let $\lfloor x \rfloor$ denote the greatest integer that is less than or equal to x, and $\lceil x \rceil$ denote the smallest integer that is greater than or equal to x.

   Compute
   $$\left\lceil \frac{75}{6} \right\rceil - \left\lfloor -\frac{75}{2} \right\rfloor .$$

   Answer (51): The expression simplifies to

$$\begin{aligned} \left\lceil \frac{75}{6} \right\rceil - \left\lfloor -\frac{75}{2} \right\rfloor &= \lceil 12.5 \rceil - \lfloor -37.5 \rfloor \\ &= 13 - (-38) \\ &= 51. \end{aligned}$$

9. In the 1980s, the area of the Great Salt Lake reached a historic high of 3200 square miles. In the early 2020s, after years of sustained drought, it fell to its lowest recorded area of 960 square miles. By what percent did the area of the lake decrease from the 1980s to the 2020s?

   Answer (70): The ratio of the area of the lake in the 2020s compared to the 1980s is

   $$\frac{960}{3200} = \frac{3}{10}, \text{ or } 30\%.$$

   So, the area of the lake decreased by 70%.

10. Page the Penguin is swimming to an island that is 8 miles away. She swims the first mile in 5 minutes, the second mile in 7 minutes, and so on, swimming each mile in 2 more minutes than the previous mile. How many minutes does it take her to reach the island?

    Answer (96): Adding the number of minutes for the 8 miles we get

    $$5 + 7 + 9 + 11 + 13 + 15 + 17 + 19.$$

    We can pair these numbers from the ends $(5+19)$, $(7+17)$, etc. There are 4 pairs and each pair has a sum of 24. So, the total sum is

    $$4 \cdot 24 = 96.$$

    Alternate Solution: The sum of the first n positive odd numbers is n^2. By adding (and then subtracting) the missing odd numbers 1 and 3, we get

    $$\begin{aligned} & 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 \\ = \ & (1 + 3 + 5 + \cdots + 17 + 19) - (1 + 3) \\ = \ & 10^2 - 2^2 \\ = \ & 100 - 4 \\ = \ & 96. \end{aligned}$$

11. Let's call a prime number *doubly prime* if the sum of its digits is also prime. For example, 7 and 41 are *doubly prime* numbers but 13 is not because the sum of its digits is 4 which is not prime. Find the seventh smallest *doubly prime* number.

    Answer (29): First, note that single-digit primes 2, 3, 5, and 7 are all doubly prime.

Among the primes in 10's (11, 13, 17, and 19) 11 is the only doubly prime because their sums of digits are all even, and among them only $1 + 1 = 2$ is prime.

The prime numbers in the 20's are 23 and 29, which are both doubly prime because $2 + 3 = 5$ and $2 + 9 = 11$ are both primes.

The first seven doubly prime numbers are

$$2, 3, 5, 7, 11, 23, \text{ and } 29.$$

Thus, the answer is 29.

12. Seven horses can eat 504 pounds of hay in 6 days. At the same rate, how many pounds of hay can 2 horses eat in 4 days?

Answer (96): If 7 horses in 6 days eat 504 pounds of hay, at this rate 1 horse in 1 day eats $\frac{504}{7\times 6} = 12$ pounds of hay. So, 2 horses in 4 days eat $2 \times 4 = 8$ times this amount which is 96 pounds of hay.

13. Stan wants to draw three towns A, B, and C on a map. He knows that the distance between A and B is 28 miles, and the distance between B and C is 67 miles. How many possible whole integer distances in miles are there between A and C?

Answer (57): By the triangle inequality, the distance between A and C (in miles) is at least $67 - 28 = 39$ and at most $67 + 28 = 95$. Note that the distance can take the smallest and largest values; the smallest value if the three cities are on a line in B-A-C order and the largest value if they are on a line in A-B-C order.

Thus, the answer is the number of integers between 39 and 95 (inclusive), which is

$$95 - 39 + 1 = 57.$$

14. A fisherman catches five pounds of tilapia and three pounds of salmon. He sells them for a total of \$133. If the price of salmon per pound is 50% more than the price of tilapia per pound, how much does he earn from the three-pound salmon?

Answer (63): Let the price of one pound of tilapia be $2a$ dollars. Then the price of one pound of salmon is $3a$ dollars. The total price of 5 pounds of tilapia and 3 pounds of salmon (in dollars) is

$$2a \cdot 5 + 3a \cdot 3 = 133.$$

Solving this, we get

$$19a = 133 \implies a = 7.$$

The price of 3 pounds of salmon is therefore

$$3a \cdot 3 = 21 \cdot 3 = 63 \text{ dollars.}$$

15. Integers m and n satisfy $3m + 4n = 29$. Find the minimum possible value of $|mn|$, the absolute value of the product mn.

Answer (8): Trying small values of n we find that $(m, n) = (7, 2)$ is a solution to the equation. If (m, n) is any solution to the equation, then we have

$$3m + 4n = 3 \cdot 7 + 4 \cdot 2.$$

From here we get

$$3m - 3 \cdot 7 = 4 \cdot 2 - 4n$$

which leads to

$$3(m - 7) = 4(2 - n).$$

Since 3 and 4 are relatively prime, $m - 7$ must be a multiple of 4, and $2 - n$ must be a multiple of 3. Letting $m - 7 = 4k$ for some integer k leads to $2 - n = 3k$ and we have

$$(m, n) = (7 + 4k, 2 - 3k).$$

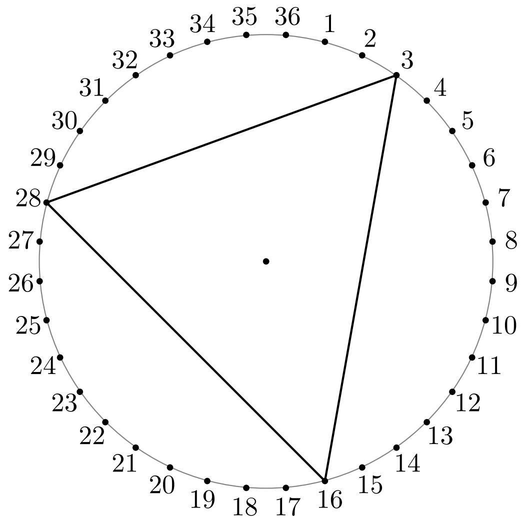
Starting with (7,2) and going up (decreasing m by 4 and increasing n by 3) and down (increasing m by 4 and decreasing n by 3), we get the following table of solutions:

m	n	$\|mn\|$
$\vdots$	$\vdots$	$\vdots$
-5	11	55
-1	8	8
3	5	15
7	2	14
11	-1	11
15	-4	60
$\vdots$	$\vdots$	$\vdots$

Finally, we note that the smallest $|mn|$ value in the table is 8.

16. Points labeled with the numbers $1, 2, 3, \ldots, 36$ are equally spaced on the circumference of a circle in that order. What is the degree measure of the smallest angle in the triangle whose vertices are the points labeled with numbers 3, 16, and 28?

Answer (55):



The degree measure of the angle tending the minor arc between the points labeled with numbers 3 and 16 is

$$\frac{1}{2} \cdot \frac{360}{36} \cdot (16 - 3) = 65.$$

The degree measure of the angle tending the minor arc between the points labeled with numbers 16 and 28 is

$$\frac{1}{2} \cdot \frac{360}{36} \cdot (28 - 16) = 60.$$

Thus, the last angle has a degree measure

$$180 - 60 - 65 = 55.$$

The answer is the smallest of these three numbers, which is 55.

17. The product of three consecutive positive integers is equal to 40 times their sum. Find the sum of the three integers.

Answer (33): Let the middle integer be n. We have

$$(n - 1)n(n + 1) = 40((n - 1) + n + (n + 1)).$$

Simplifying the equation we get

$$(n^2 - 1)n = 120n.$$

Since n is not zero, we can divide both sides of the equation by n, leading to

$$n^2 - 1 = 120 \implies n^2 = 121.$$

The positive solution for n is $n = 11$, and the sum of the three integers is

$$(n-1) + n(n+1) = 3n = 33.$$

18. The real numbers a and b satisfy $a+b = 3$ and $a^2+b^2 = 7$. Find the value of a^4+b^4.

Answer (47): First, squaring $a + b$ we get

$$\begin{aligned} (a+b)^2 &= 3^2 \\ (a^2+b^2) + 2ab &= 9 \\ 7 + 2ab &= 9 \\ ab &= 1. \end{aligned}$$

Now, squaring $a^2 + b^2$ and using $ab = 1$ we obtain

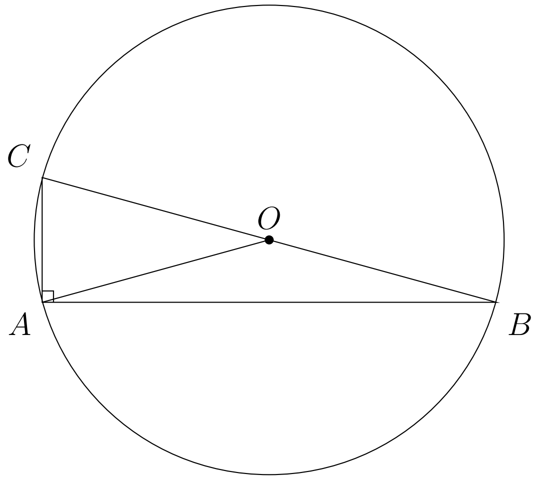
$$\begin{aligned} (a^2+b^2)^2 &= 7^2 \\ a^4 + b^4 + 2(ab)^2 &= 49 \\ a^4 + b^4 + 2 \cdot 1^2 &= 49 \\ a^4 + b^4 &= 47. \end{aligned}$$

19. A triangle has sides of lengths 9, 40, and 41. The ratio of its circumradius to its inradius can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer (49): Note that since $9^2 + 40^2 = 41^2$, this is a right triangle.

Because $\angle BAC = 90°$, we have $\overset{\frown}{BC} = 2\angle BAC = 180°$. This means $\overline{BC}$ is a diameter of the circumcircle of $\triangle ABC$. It follows that the circumcenter is the midpoint of $\overline{BC}$ and the circumradius is

$$R = \frac{BC}{2} = \frac{41}{2}\cdot$$



To find the inradius recall that in any triangle $\triangle ABC$, we have sr is the area of $\triangle ABC$, where s is the semi-perimeter of the triangle and r is the inradius. For our triangle ABC, the area is $\frac{9 \cdot 40}{2} = 180$ and $s = \frac{9+40+41}{2} = 45$. Hence, we obtain

$$45r = 180 \implies r = 4.$$

Combining these, we conclude that the ratio of the circumradius to the inradius is

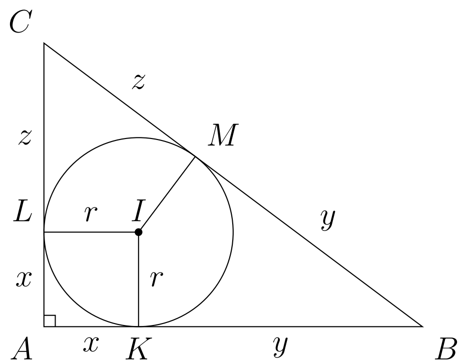
$$\frac{R}{r} = \frac{41/2}{4} = \frac{41}{8}.$$

Hence, the answer is $41 + 8 = 49$.

Remark: In a right triangle ($\triangle ABC$ with $\angle A = 90°$), the inradius can be found in another way as follows.

Let K, L, M be the feet of the altitudes from I to the sides of the triangle. Let $x = AK = AL$, $y = BK = BM$, and $z = CL = CM$. Then observe that $AKIL$ is a square. This is because three of its angles ($\angle A$, $\angle K$, and $\angle L$) are right angles which forces the fourth one ($\angle LIK$) to be a right angle as well. So it is a rectangle. Moreover the side lengths AK and AL are the same, so it is not just any rectangle but a square indeed.

Thus, we conclude that $x = r$.



To find $r = x$, let $a = BC$, $b = AC$, $c = AB$, and $s = \frac{a+b+c}{2}$ (half perimeter). Then we have

$$\begin{aligned} x + y &= c \\ y + z &= a \\ z + x &= b \end{aligned}$$

Adding these three equations and dividing by 2, we get

$$x + y + z = s.$$

Subtracting this from the $y + z = a$ equation gives us

$$x = s - a.$$

Since $x = r$, we conclude that $r = s - a$ (where $\angle A$ is the right angle).

Note that in our triangle, $s = \frac{9+40+41}{2} = 45$ and $a = 41$. So, $r = s - a = 4$, as before.

20. The numbers 5 and 6 are written on a whiteboard. Every minute, Edwin writes a number on the board that is either twice a number on the board or the sum of two numbers on the board. If $S = \{5, 6, 10, 11, 12, \ldots\}$ is the set of numbers that can be written on the board, how many elements in S are at most 100?

Answer (90): Note that the numbers that can be written on the board are positive integers of the form $5m + 6n$ where m and n are non-negative integers.

It follows that if a number k can be written on the board, then so can $k + 5$. So, let's list the numbers in groups of 5 in the following table, and in each column circle the first number that can be written on the board. We obtain 6 for the first column, 12 for the second, 18 for the third, 24 for the fourth, and 5 for the fifth:

1	2	3	4	(5)
(6)	7	8	9	10
11	(12)	13	14	15
16	17	(18)	19	20
21	22	23	(24)	25
⋮	⋮	⋮	⋮	⋮
96	97	98	99	100

Any number that is below a circled number can be obtained by adding 5's to that number, hence it can be written on the board as well. Also, since the circled numbers are the first ones in their columns that can be written on the board, none of the numbers above them can be written on the board. Counting these by column, we get $1+2+3+4 = 10$ of them. These are 1, 2, 3, 4, 7, 8, 9, 13, 14, and 19. Therefore, the remaining $100 - 10 = 90$ numbers can be written on the board.

Alternate Solution: By the Chicken McNugget Theorem, given two relatively prime positive integers a and b, the largest number that cannot be written as a sum of a's and b's is $ab - a - b$. Applying this to $a = 5$ and $b = 6$ we find that the largest number that cannot be written on the board is $5 \times 6 - 5 - 6 = 19$. Checking numbers less than 19, we find only 9 numbers that can be written on the board:

$$5,\ 6,\ 10,\ 11,\ 12,\ 15,\ 16\ (5+5+6),\ 17\ (5+6+6),\ \text{and } 18.$$

So the remaining $19 - 9 = 10$ numbers cannot be written on the board. Overall, out of 100 numbers, $100 - 10 = 90$ of them can be written on the board.

21. A box contains four apples, six oranges, and a pear. Preston reaches into the box and selects two of the 11 fruits uniformly at random, without replacement. Let $\frac{m}{n}$ be the probability that he selects an apple and an orange, where m and n are relatively prime positive integers. Find $m + n$.

Answer (79): The probability of Preston selecting an apple followed by an orange is

$$\frac{4}{11} \times \frac{6}{10} = \frac{12}{55}.$$

But note that he can also select an orange followed by an apple with the same probability. So, the desired probability is twice this, which is $\frac{24}{55}$. The answer is

$$24 + 55 = 79.$$

22. The first three terms of an arithmetic sequence are 1.85, 2.74, and 3.63 in that order. Find the first integer term in this sequence.

Answer (33): The common difference of the sequence is $2.74 - 1.85 = 0.89$. So, the kth term of the sequence can be written as

$$a_k = 1.85 + 0.89(k - 1) = 0.96 + 0.89k.$$

Let a_n be the first integer term in this sequence. Multiplying

$$a_n = 0.96 + 0.89n$$

by 100, we get

$$100a_n = 96 + 89n$$

which must be a multiple of 100. Adding $300 - 100n$ to this, we see that $396 - 11n$ must also be a multiple of 100. But this is $11(36 - n)$. Since 11 and 100 are relatively prime, $36 - n$ must be a multiple of 100. The smallest positive integer n satisfying this is 36.

Our answer is

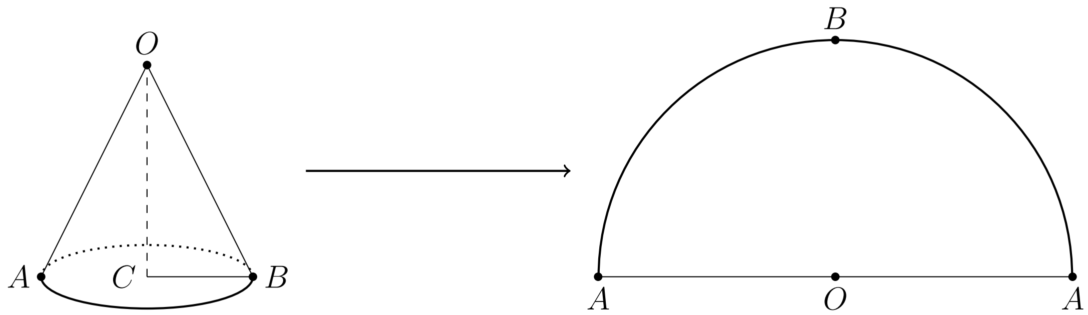
$$a_{36} = 0.96 + 0.89 \times 36 = 0.96 + 32.04 = 33.$$

Remark: The value of a_{36} can also be calculated as follows:

$$\begin{aligned} a_{36} &= 0.96 + 0.89 \times 36 \\ &= 0.96 + (1 - 0.11) \times 36 \\ &= 0.96 + 36 - 3.96 \\ &= 36 - 3 \\ &= 33. \end{aligned}$$

23. The lateral surface of a right cone can be unrolled into a semicircle of area 96π. What is the height of the cone?

Answer (12):



Let's first calculate the radius of the semicircle BO which is also the slant height of the cone:

$$\frac{\pi \cdot BO^2}{2} = 96\pi \implies BO^2 = 192 \implies BO = 8\sqrt{3}.$$

The length of the curved side of the semicircle ($A\overset{\frown}{B}A$ on the right figure) is the same as the circumference of the cone's base. So we have

$$\frac{2\pi \cdot BO}{2} = 2\pi \cdot BC.$$

It follows that

$$BC = \frac{BO}{2} = 4\sqrt{3}.$$

Finally, either by using Pythagorean Theorem on $\triangle BCO$ with $\angle BCO = 90°$, or by observing that BC is half of OB, we see that this is a 30-60-90 triangle, and deduce that

$$OC = \sqrt{3}BC = 12.$$

24. Let p and q be prime numbers satisfying

$$p^3 - pq^2 = 42024.$$

Find the sum of the digits of the product pq.

Answer (8): Taking the common factor out, we get

$$p(p^2 - q^2) = 42024.$$

The prime factorization of 42024 is $2^3 \times 3 \times 17 \times 103$. Since p is a prime dividing 42024, it must be one of these prime factors: 2, 3, 17, or 103. Also,

$$p^3 = 42024 + pq^2 > 42024 > 8000 \implies p > 20.$$

Therefore, p can only be 103. Plugging in $p = 103$ we get

$$103^2 - q^2 = 408.$$

Solving this equation gives us

$$q^2 = 10609 - 408 = 10201 \implies q = 101.$$

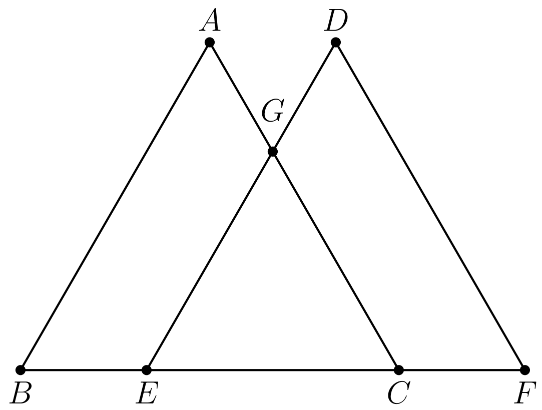
Finally,

$$pq = 103 \times 101 = 10403,$$

and our answer is

$$1 + 4 + 3 = 8.$$

25. In the figure below, $\triangle ABC$ and $\triangle DEF$ are equilateral triangles with side length 2. They are placed so that points B, E, C, and F lie on the same line. The segments $\overline{AC}$ and $\overline{DE}$ intersect at point G. The area of the union of the two triangles is the concave pentagon $BAGDF$ whose area is $6 - 2\sqrt{3}$. If $BF = a - \sqrt{b}$ for some positive integers a and b, what is $a + b$?



Answer (18): Observe that $\triangle GEC$ is also an equilateral triangle. Let $EC = x$. Then the area of $BAGDF$ is the sum of the areas of $\triangle ABC$ and $\triangle DEF$ minus the area of $\triangle GEC$. Using this we get

$$6 - 2\sqrt{3} = 2 \cdot \frac{2^2\sqrt{3}}{4} - \frac{x^2\sqrt{3}}{4}.$$

Solving this, we obtain

$$\begin{aligned} & x^2 = 16 - 8\sqrt{3} = 4(4 - 2\sqrt{3}) = 4(\sqrt{3} - 1)^2 \\ \implies \quad & x = 2\sqrt{3} - 2. \end{aligned}$$

Then

$$\begin{aligned} BF &= BC + EF - EC \\ &= 4 - x. \end{aligned}$$

Plugging in $x = 2\sqrt{3} - 2$ gives

$$BF = 6 - 2\sqrt{3} = 6 - \sqrt{12}.$$

The answer is $6 + 12 = 18$.