Pi Math Contest Euler Division

2023 Solutions

The problems and solutions in this contest were proposed by:

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Solutions

1. Compute
$$2 \times 4 - 2.$$

Answer (6): Doing multiplication first, we get
$$\begin{aligned} & (2 \times 4) - 2 \\ = \ & 8 - 2 \\ = \ & 6. \end{aligned}$$

2. What is the value of
$$\frac{1+2+3+4+5}{3} \ ?$$

Answer (5):
$$1 + 2 + 3 + 4 + 5 = 15.$$

So, the fraction is
$$\frac{15}{3} = 5.$$

3. Allie will be 7 years old in two years. How old was Allie two years ago?

   Answer (3): Allie is $7-2=5$ years old now. So, two years ago she was $5-2=3$ years old.

4. Find the value of
$$100-99+98-97+96-95.$$

   Answer (3): Pairing up the numbers, we get
$$(100-99)+(98-97)+(96-95).$$
   Because the difference between the numbers in each pair is 1, the sum is
$$1+1+1=3.$$

5. What is 30% of 30?

   Answer (9): 10% of 30 is a tenth of 30 which is 3.
   30% of 30 is three times this. So, it is $3\times 3=9$.

6. What is
$$(2.5\times 4)-(2\times 4.5)\ ?$$

   Answer (1): First, 2.5×4 is twice 4 added to half of 4, so it is $8+2=10$.
   Next, 2×4.5 is four times 2 added to half of 2, which is $8+1=9$.
   The answer is $10-9=1$.

7. What is the units digit of the product: $12\times 13\times 14$?

   Answer (4): We want to find only the units digit of the product. Since the tens digits of the factors will not contribute to the units digit of the product, we can ignore them, and get
$$2\times 3\times 4=24$$
   whose units digit is 4.

8. The hour hand on a clock points at the number 3. After the hour hand moves 120° clockwise, which number will the hour hand point at?

   Answer (7): Note that the hour hand makes one complete rotation of 360° in 12 hours. So, it moves by 120° in one-third of this time, which is 4 hours. After 4 hours from 3 o'clock, it will be 7 o'clock.

9. What is $\left(2+\frac{1}{3}\right)\left(3+\frac{1}{2}\right)$, rounded to the nearest whole number?

   Answer (8): Adding the expressions in parentheses gives

   $$\frac{7}{3}\times\frac{7}{2}=\frac{49}{6}.$$

   As a mixed number, this is $8\frac{1}{6}$; rounded to the nearest integer, it is 8.

   Alternate Solution: Expanding the expression we get

   $$\begin{aligned}\left(2+\frac{1}{3}\right)\left(3+\frac{1}{2}\right) &= 2\times\left(3+\frac{1}{2}\right)+\frac{1}{3}\times\left(3+\frac{1}{2}\right)\\ &= (2\times 3)+\left(2\times\frac{1}{2}\right)+\left(\frac{1}{3}\times 3\right)+\left(\frac{1}{3}\times\frac{1}{2}\right)\\ &= 6+1+1+\frac{1}{6}\\ &= 8\frac{1}{6}.\end{aligned}$$

   Once again, rounding this to the nearest integer, we get 8.

10. In triangle $\triangle ABC$, let $\overline{AD}$ be the altitude from A. If $BC=8$ and the area of $\triangle ABC$ is 24, what is AD?

    Answer (6): Since the area is half of the product of the length of the base and the length of the height, this means that this product is equal to $2\times 24=48$. So, the height has length $\frac{48}{8}=6$ inches.

11. How many ways are there to arrange the letters in the word GEESE, so that no two E's are next to each other?

    Answer (2): The E's must be the first, third, and fifth letters in order to not be next to each other. There are then 2 ways to place the other two letters, which leads to 2 arrangements. They are EGESE and ESEGE.

12. What is the positive difference between the 47th positive odd number and the 43rd positive even number?

    Answer (7): The first odd number is 1 less than twice 1. The second odd number is 1 less than twice 2, and so on. So, the 47th odd number will be 1 less than twice 47, which means that it is $2 \times 47 - 1 = 93$.

    For even numbers, the first even number is 2 (twice 1), the second one is 4 (twice 2), and so on. So, the 43rd even number is $2 \times 43 = 86$.

    The difference is $93 - 86 = 7$.

13. How many positive integers are divisors of 36?

    Answer (9): Each factor of 36 can be paired with another so that their product is 36. Using this idea, we get the following pairs of factors of 36:

$$36 = 1 \times 36$$
$$36 = 2 \times 18$$
$$36 = 3 \times 12$$
$$36 = 4 \times 9$$
$$36 = 6 \times 6$$

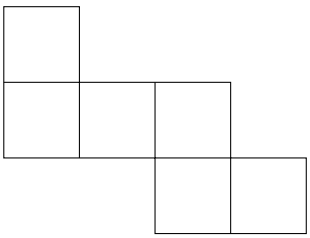
    Therefore, there are 9 factors of 36:

$$1,\ 2,\ 3,\ 4,\ 6,\ 9,\ 12,\ 18,\ 36.$$

14. In how many ways could four different coins be flipped such that two of them land on heads and the other two land on tails? One example is *HHTT*, where the first two coins land on heads, and the last two coins land on tails.

    Answer (6): Let's number the coin placements from left to right as 1, 2, 3, and 4. The placements of the two H's uniquely determine where the two tails are. There are 6 possibilities for the placement of the two H's: 12, 13, 14, 23, 24, and 34. Therefore, the answer is 6. The possible sequences are: *HHTT*, *HTHT*, *HTTH*, *THHT*, *THTH*, and *TTHH*.

15. The following figure is formed by squares. The perimeter is 28. What is the units digit of its area?



Answer (4): The perimeter consists of 14 line segments, each of which has length equal to the side length of the square. Thus, the side length of the square is $\frac{28}{14} = 2$. There are 6 squares, each with area 4. So, the area of the whole figure is $6 \times 4 = 24$, and its units digit is 4.

16. A classroom has a row of six light switches each of which can be either ON or OFF. How many ways are there for these six switches such that at least five of them are ON?

Answer (7): Note that the number of ON switches must be either 5 or 6.

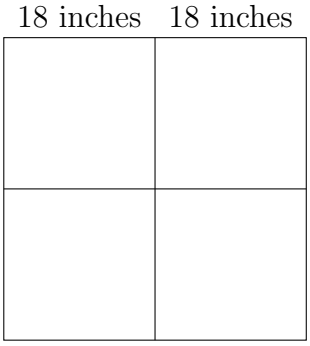
If 5 switches are ON, then only one switch is OFF and this can be switch number 1, 2, 3, 4, 5, or 6. This gives us 6 ways.

There is only 1 way if 6 switches are ON.

Therefore, the total number of ways is $6 + 1 = 7$.

17. Daniel has a large supply of 18-inch by 18-inch squares of origami paper. He places the greatest number of his origami papers that can fit into a 1-yard by 1-yard square without overlapping them. What is the number of origami papers he places? There are 3 feet in a yard and 12 inches in a foot.

Answer (4): Note that 1 yard is 3 feet which is the same as $3 \times 12 = 36$ inches. Since the origami papers have side lengths of 18 inches, Daniel can fit 2 of these on each side of the 1-yard by 1-yard square, as shown in the diagram below. Therefore, he can fit 4 of these inside the square by dividing it into 4 smaller squares of equal size.



18. The inhabitants of the Alpha Star have three different currencies: zips, zags, and zugs. If one zug is four zags and two zags is five zips, how many zugs is twenty zips equal to?

Answer (2): Since

$$5 \text{ zips} = 2 \text{ zags}$$

multiplying both sides by 4, we find

$$20 \text{ zips} = 8 \text{ zags}.$$

Also,

$$4 \text{ zags} = 1 \text{ zug}.$$

So, doubling both sides we get

$$8 \text{ zags} = 2 \text{ zugs}.$$

We deduce that

$$20 \text{ zips} = 8 \text{ zags} = 2 \text{ zugs}.$$

Therefore, the answer is 2.

19. Avi, Beyonce, and Chi-Chi each have 8 tokens. Avi gives half of his tokens to Beyonce. Then, Beyonce gives half of her tokens to Chi-Chi. Finally, Chi-Chi gives half of her tokens to Avi. In the end, Avi has how many more tokens than Beyonce?

Answer (5): Let's keep track of the number of tokens each person has using the following table:

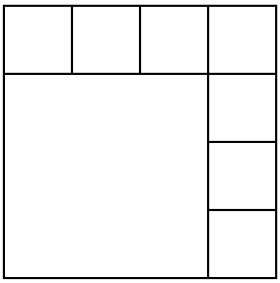
	Avi	Beyonce	Chi-Chi
Initial:	8	8	8
After Avi gives half to Beyonce:	4	12	8
After Beyonce gives half to Chi-Chi:	4	6	14
After Chi-Chi gives half to Avi:	11	6	7

For example, in the first step, Avi gives half of his 8 tokens (or 4 tokens) to Beyonce, so Avi is left with 4 tokens and Beyonce has $8 + 4 = 12$ tokens, and so on.

In the end, Avi has 11 tokens while Beyonce has 6. So, the difference is 5.

20. A 12×12 square is covered by a 9×9 square and several 3×3 squares, without overlap. How many 3×3 squares are used?

Answer (7):



A 12×12 square has area 144, whereas a 9×9 square has an area of 81.

It means that we need $144 - 81 = 63$ more square units of area. Each 3×3 square has an area of 9. So, we need to add

$$\frac{63}{9} = 7$$

3×3 squares.

21. A plane starts taking off at sea level. For every 30 miles it travels, its altitude increases by $\frac{1}{10}$ of a mile until it reaches its highest altitude. After that, its altitude goes down by $\frac{1}{10}$ of a mile every 30 miles until it lands at sea level. If the plane travels a total of 1200 miles before arriving at sea level, what is the highest altitude it reaches, in miles?

Answer (2): Since the plane's altitude is increasing and decreasing at the same rates, the plane reaches its highest altitude in 600 miles (half the distance covered).

We are given that:

It travels 30 miles as its altitude increases by $\frac{1}{10}$ miles.

Scaling by 10, we find that:

It travels 300 miles as its altitude increases by 1 mile.

Finally, scaling by 2, we obtain that

It travels 600 miles as it reaches its highest altitude of 2 miles.

Therefore, the answer is 2.

22. Bessie has between 10 and 100 cows. She arranges them into groups. When she puts them into groups of 8, five cows are left over. When she puts them into groups of 9, six cows are left over. How many cows are left over when she puts them into groups of 10?

Answer (9): Note that both remainders (5 and 6) are 3 less than their respective size of groups (8 and 9). So, if Bessie had 3 more cows the number of her cows would be a multiple of both 8 and 9.

The only multiple of 8 and 9 between 10 and 100 is $8 \times 9 = 72$. So, the actual number of her cows is $72 - 3 = 69$. If she puts them into groups of 10, there will be 9 cows left over.

23. Buzz is a robot who weighs 20 pounds. He has a large collection of identical two-pound toys and identical three-pound toys. He wants to select some of these toys so that their total weight is the same as his own weight. In how many can he do this? For example, one possibility is to use 6 three-pound toys and 1 two-pound toy. The order of the toys in the selection is not important.

Answer (4): We want to write 20 as a sum of 2's and 3's where the order of the numbers is not important. Note that 20 is even, therefore we must have an even number of 3's in the collection. The number of 3's can be 0, 2, 4, or 6. Note that we cannot have more than 6 of the 3's because that would exceed 20. Each of these cases corresponds to one case as shown in the table below. So, the answer is 4.

Number of 3's	Number of 2's
0	10
2	7
4	4
6	1

24. A bank has been robbed. There are four suspects numbered:

1 (Andy), 2 (Beatrice), 3 (Cindy), and 4 (Danny).

Only one of them is guilty. The suspects give the following statements:

Andy: "I and Danny are innocent."
Beatrice: "I am guilty."
Cindy: "Beatrice is innocent."
Danny: "Beatrice is guilty."

If exactly one of them is lying (the lying person isn't necessarily the guilty person), what is the sum of the suspect number of the lying person and the suspect number of the guilty person?

Answer (5): The statements of Beatrice and Danny are the same. Since both of them cannot be lying, they must both be telling the truth. It means that Beatrice (2) is indeed guilty and Cindy (3) is lying.

So, the answer is $3 + 2 = 5$.

25. A *palindrome* is a number that reads the same forward and backward. For example, 505 is a palindrome. How many 3-digit palindromes are divisible by 11?

Answer (8): Let $\overline{aba}$ be a three-digit palindrome that is a multiple of 11. We can write this as

$$100a + 10b + a = 101a + 10b.$$

Note that $99a+11b$ is also a multiple of 11. Subtracting the two, we obtain another multiple of 11:

$$2a - b.$$

Since a and b are digits, $2a - b$ must be larger than -10 and less than 20. The only multiples of 11 in this interval are 0 and 11. So, $2a - b$ must be either 0 or 11.

The case $2a - b = 0$ gives the following solutions:

$$121,\ 242,\ 363,\ 484,$$

whereas $2a - b = 11$ leads to solutions:

$$616,\ 737,\ 858,\ 979.$$

Overall, there are $4 + 4 = 8$ solutions.