Pi Math Contest Fermat Division

2022 Solutions

The problems and solutions in this contest were proposed by:

Ali Gurel, Hanna Chen, Isaac Li, Mason Fang, Nikhil Thakur, Preston Fu, Sanya Badhe, and Tiger Che.

Solutions

1. Find the value of $2 + 0 \times 2 - 2$.

   Answer (0): We need to do the multiplication before the addition. We get

   $$2 + (0 \times 2) - 2 = 2 + 0 - 2 = 2 - 2 = 0.$$

2. Compute

   $$\frac{333 - 33 + 3 \times 33}{3^3 - 3 - 3}.$$

   Answer (19): We directly compute to get

   $$\frac{300 + 99}{27 - 6} = \frac{399}{21} = 19.$$

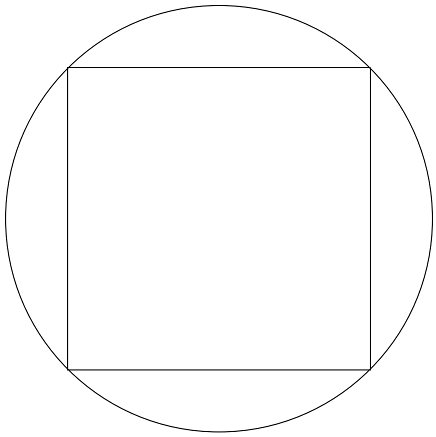
3. How many ways are there to select one marble from a bag of 20 unique red marbles, 29 unique blue marbles, and 38 unique green marbles?

   Answer (87): There are $20 + 29 + 38 = 87$ marbles to choose one marble from.

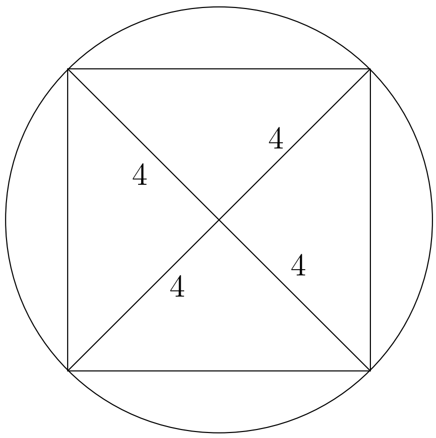
4. How many total sides are there among 15 triangles and 4 squares, if no side is shared by more than one shape?

   Answer (61): There are $15 \cdot 3 = 45$ sides among 15 triangles, and $4 \cdot 4 = 16$ sides among 4 squares. In total, there are $45 + 16 = 61$ sides.

5. A square is inscribed in a circle, as shown below. If the area of the circle is 16π, what is the area of the square?

   Answer (32):

   Since the area of the circle is $4^2\pi$, its radius is 4. Drawing the two diagonals of the square, we get four congruent isosceles right triangles, with legs 4. Each of these triangles has area $\frac{4 \cdot 4}{2} = 8$, so the area of the square is $4 \cdot 8 = 32$.

6. Four times a number plus three is equal to seven times the number minus twelve. What is the number?

   Answer (5): Let the number be x. We know $4x + 3 = 7x - 12$, so $3x = 15$, which gives that $x = 5$.

7. Five friends go to a local grocery store to buy bottles of water. Each bottle of water costs \$1.50, and each person has \$10.00. If they combine their money, what is the maximum number of bottles they can buy?

   Answer (33): Altogether, they have \$50.00. If they had \$150, they could buy 100 bottles of water. With one third that money, they can buy only one third of 100, or 33 bottles of water.

8. Bamboo grows at a rate of 1.7 inches per day. Tanya buys a bamboo stalk which currently stands 1 foot and 3 inches tall. If Tanya is 5 feet and 6 inches tall, then in how many days will the bamboo stalk grow to her current height? Recall that 1 foot equals 12 inches.

   Answer (30): Tanya's height is $5 \cdot 12 + 6 = 66$ inches. The bamboo stalk is currently $1 \cdot 12 + 3 = 15$ inches tall. So, to reach Tanya's height, it needs to grow by $66 - 15 = 51$ inches, which would take $\frac{51}{1.7} = 30$ days.

9. Compute the sum

   $$\gcd(1, 12) + \gcd(2, 12) + \cdots + \gcd(11, 12) + \gcd(12, 12),$$

   where $\gcd(a, b)$ denotes the greatest common divisor of a and b.

   Answer (40): Compute each of the 12 terms we get:

   $$1 + 2 + 3 + 4 + 1 + 6 + 1 + 4 + 3 + 2 + 1 + 12 = 40.$$

10. Annie and Mercie each have 32 pencils. To start, Annie gives Mercie half of her pencils. Since they are overly kind to each other, the person who has more pencils will always give half of her pencils to the other. This continues until the person with more pencils has an odd number of pencils. At that point, how many pencils will Mercie have?

    Answer (43): Here is a chart detailing the number of pencils after each exchange:

Annie	32	16	40	20	42	21
Mercie	32	48	24	44	22	43

So, Mercie will end up with 43 pencils.

11. Simplify:

$$\frac{2^{2^{0^2}} + 2^{2^0} + 2^2 + 2}{2^{0^{2^2}} + 2^{0^2} + 2^0 + 2}.$$

Answer (2): Note that $2^2 = 4$ and $2^{2^{0^2}} = 2^{2^0} = 2^1 = 2$. Also, $2^{0^{2^2}}$ and 2^{0^2} are both equal to 2^0, which is 1. Plugging these in, we get

$$\frac{2+2+4+2}{1+1+1+2} = \frac{10}{5} = 2.$$

12. Let N be the number of three-digit positive integers that have at least two distinct digits. Find the last two digits of N.

Answer (91): We use complementary counting. There are $9 \cdot 10 \cdot 10 = 900$ three digit integers, and exactly 9 are comprised of only one digit, namely $111, 222, \ldots, 999$. Therefore, $N = 900 - 9 = 891$, and the answer is 91.

13. A square and a regular hexagon have equal perimeters. If the area of the hexagon is $24\sqrt{3}$, what is the area of the square?

Answer (36): Let s and a be the side lengths of the regular hexagon and square, respectively. Split the regular hexagon into 6 equilateral triangles (each with side length s). The area of one equilateral triangle is

$$\frac{s^2\sqrt{3}}{4} = \frac{24\sqrt{3}}{6} = 4\sqrt{3}.$$

.

Solving this, we find that $s = 4$. Since the perimeters of the hexagon and square are the same, we have

$$6s = 4a \implies a = 6.$$

Finally, the area of the square is $a^2 = 36$.

14. Suppose 80% of A is 56% of B, and 45% of B is 27% of C. What percent of C is 150% of A?

Answer (63): Writing the first two statements in equations with fractions, we have $\frac{4}{5}A = \frac{14}{25}B$ and $\frac{9}{20}B = \frac{27}{100}C$. Solving for A in the first equation and B in the second equation gives $A = \frac{7}{10}B$ and $B = \frac{3}{5}C$, so $A = \frac{21}{50}C$. So, 150% of A is $\frac{3}{2}A = \frac{63}{100}C = 63\%$ of C.

Alternate Solution: We will scale the given equations to write all the comparisons in one line.

First let's simplify the first equation:

$$80\% \text{ of } A = 56\% \text{ of } B \quad \Rightarrow \quad 10\% \text{ of } A = 7\% \text{ of } B.$$

Similarly, we can simplify the second equation:

$$45\% \text{ of } B = 27\% \text{ of } C \quad \Rightarrow \quad 5\% \text{ of } B = 3\% \text{ of } C.$$

In the simplified equations, we have 7% of B and 5% of B. To make these equal, we scale them to 35% of B. This gives us

$$50\% \text{ of } A = 35\% \text{ of } B = 21\% \text{ of } C.$$

Finally, since we want 150% of A, we triple this equation and reach

$$150\% \text{ of } A = 105\% \text{ of } B = 63\% \text{ of } C.$$

15. Every day, a sprinkler first turns on at 9 AM. To save water, Alice turns it off every 50 minutes, at 9:50 AM, 10:40 AM, and so on. To water the plants, Bob turns it on every 60 minutes, at 10 AM, 11 AM, and so on. They continue playing like this until 2 PM. At 2 PM, the sprinkler turns off for the rest of the day. The fraction of the 24 hours of the day the sprinkler is on can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer (53): Let's follow the procedure. The sprinkler is on from 9 to 9:50, from 10 to 10:40, from 11 to 11:30, from 12 to 12:20, and from 1 to 1:10. It is on for a total of

$$50 + 40 + 30 + 20 + 10 = 150 \text{ minutes, or } 2.5 \text{ hours.}$$

This is $\frac{2.5}{24} = \frac{5}{48}$ of the day, so the answer is $5 + 48 = 53$.

16. Three different primes have a mean of 58. What is the largest possible value of the median of these three primes?

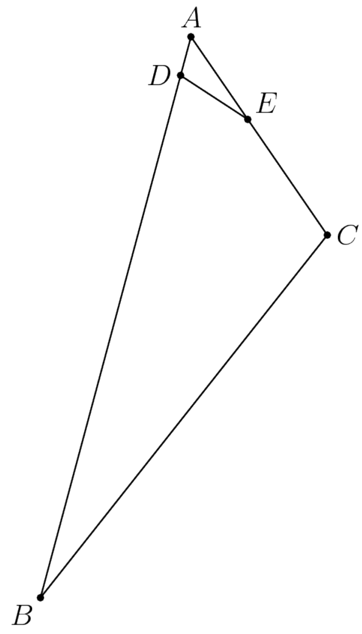
    Answer (83): The sum of the three primes is $3 \times 58 = 174$. Since the sum is even, one of the primes must be even as the sum of three odd numbers is odd, and the only even prime is 2. Then the remaining two primes have a sum of 172. To make the middle prime as large as possible, we would like the two odd primes to be as close as possible; this is achieved with the primes 83 and 89, because $83 + 89 = 172$. Note that 85 and 87 are not primes. Thus, the largest possible value of the median prime is 83.

17. A two-digit number is 33 more than a positive perfect square and 31 less than another positive perfect square. What is it?

    Answer (69): The difference between the two perfect squares (say x^2 and y^2) is $33 + 31 = 64$, so $(x + y)(x - y) = 64$. Since x and y are positive integers, it follows that the only possibilities for $(x + y, x - y)$ are $(32, 2)$ and $(16, 4)$. We can solve to find that the corresponding (x, y) are $(17, 15)$ and $(10, 6)$. These would give answers of $15^2 + 33 = 258$ and $6^2 + 33 = 69$. Only the latter has two digits.

    Alternate Solution: After noting that we are looking for two perfect squares that differ by 64, we can try to find the larger of these. Since it is 64 more than a positive perfect square it must be more than 64. Also, it is 31 more than our two-digit number, so it can be at most 130. The perfect squares between 65 and 130 are 81, 100, and 121. Subtracting 64 from these, we see that only $100-64 = 36$ is a perfect square (while $81 - 64 = 17$ and $121 - 64 = 57$ are not). Hence, the two perfect squares are 36 and 100, and our two-digit number is $33+36 = 100-31 = 69$.

18. In the following diagram, $AD = 2, AE = 5, DE = 4$, and $EC = 7$. Also, $\angle AED = \angle ABC$. What is the sum of the lengths BD and BC?



Answer (52): Since the two triangles share a vertex angle, and $\angle AED = \angle ABC$, triangles $\triangle AED$ and $\triangle ABC$ are similar. Thus, we have

$$\frac{AD}{AC} = \frac{DE}{BC} = \frac{AE}{AB}.$$

Plugging in our given conditions gives

$$\frac{2}{12} = \frac{4}{BC} = \frac{5}{AB}.$$

From these equations we find that $BC = 6 \cdot 4 = 24$, and $AB = 6 \cdot 5 = 30$. Thus, $BD = 30 - 2 = 28$, and the answer is

$$BD + BC = 28 + 24 = 52.$$

19. Each square is replaced with an addition symbol (+) or a multiplication symbol ($\times$) in the expression

$$1 \square 2 \square 3 \square 4 \square 5.$$

If the greatest possible result when the arithmetic operations are carried out is M and the least possible result is m, find the remainder when $M - m$ is divided by 100.

Answer (7): For the maximum, it is better to fill in the last three boxes with $\times$, because

$$2 \times 3 > 2 + 3, \quad 3 \times 4 > 3 + 4, \text{ and } 4 \times 5 > 4 + 5.$$

Since adding 1 increases the value while multiplying by 1 doesn't, we have

$$M = 1 + 2 \times 3 \times 4 \times 5 = 121.$$

By similar logic,

$$m = 1 \times 2 + 3 + 4 + 5 = 14.$$

Finally, $M - m = 121 - 14 = 107$, and the answer is 7.

20. Ali wants to distribute 13 identical candies among Mason, Hanna, Tiger, and Jonathan, while satisfying all of the following conditions:

- Mason will get an odd number of candies.
- Hanna will get more candies than Mason.
- Tiger will get more candies than Mason and Hanna combined.
- Jonathan will get more candies than Tiger.

How many ways are there for Ali to distribute the candies?

Answer (1): Since Mason gets an odd number of candies, he must have at least 1 candy. Then, Hanna must have more, so at least 2; Tiger must have more than Mason and Hanna combined, so at least 4; Jonathan must have more than Tiger, or at least 5.

Giving everyone these minimums, we have so far distributed $1 + 2 + 4 + 5 = 12$ candies and have only 1 candy left.

If the last candy goes to anyone but Jonathan, then the chain of conditions above will force Jonathan to have an extra candy as well. So, we can give the extra candy only to Jonathan, giving us the only distribution as $1, 2, 4, 6$ candies given to Mason, Hanna, Tiger, and Jonathan respectively. Thus, the answer is 1.

21. Compute the sum of the distinct prime factors of

$$2^{15} + 2^8 - 2^7 - 1.$$

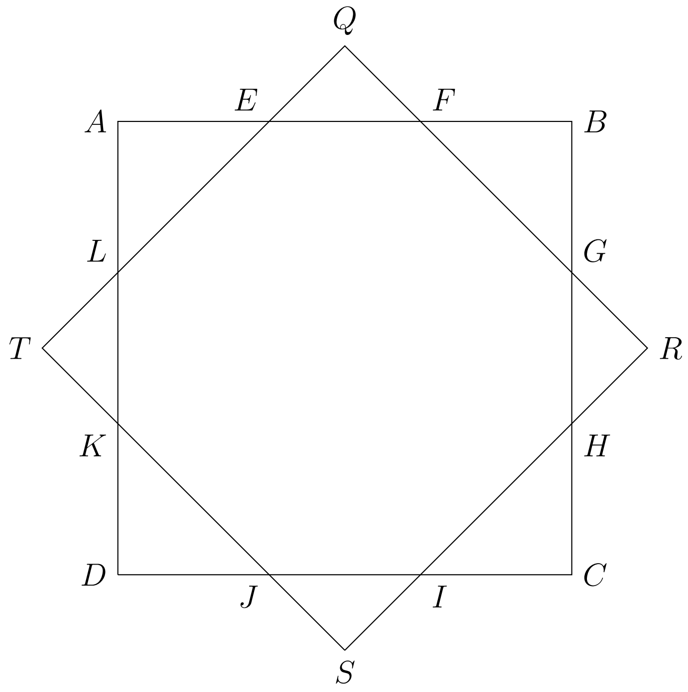
Answer (68): This expression can be factored as:

$$\begin{aligned}(2^8 - 1)(2^7 + 1) &= (2^4 - 1)(2^4 + 1)(2^7 + 1) \\ &= 15 \cdot 17 \cdot 129 \\ &= 3^2 \cdot 5 \cdot 17 \cdot 43.\end{aligned}$$

Since 3, 5, 17, and 43 are all prime numbers, the sum of the distinct prime factors is

$$3 + 5 + 17 + 43 = 68.$$

22. In the diagram below, $ABCD$ is a unit square. E, F, G, H, I, J, K, L divide each side of the square into three equal segments, and lie on square $QRST$. The ratio of the area of the intersection of these two squares to the area of the union of these two squares is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Answer (17): We will denote the area of a polygon using brackets around its vertices. For example $[QRST]$ denotes the area of the square $QRST$ while $[AEL]$ denotes the area of $\triangle AEL$.

We first consider the intersection of the two squares. This is the unit square $ABCD$, except for the four triangles $\triangle AEL$, $\triangle BFG$, $\triangle CHI$, and $\triangle DJK$ that lie outside the second square $QRST$. So, its area is

$$\begin{aligned} & [ABCD] - 4 \cdot [AEL] \\ = \ & 1 - 4 \cdot \frac{\left(\frac{1}{3}\right)^2}{2} \\ = \ & 1 - \frac{2}{9} = \frac{7}{9}. \end{aligned}$$

Next, we consider the union of the two squares. This is the unit square, as well as the four triangles $\triangle QEF$, $\triangle RGH$, $\triangle SIJ$, and $\triangle TKL$ that lie outside the unit square. So, its area is

$$\begin{aligned} & [ABCD] + 4 \cdot [QEF] \\ = \ & 1 + 4 \cdot \frac{\left(\frac{1}{3\sqrt{2}}\right)^2}{2} \\ = \ & 1 + \frac{1}{9} = \frac{10}{9}. \end{aligned}$$

Thus, the desired ratio is

$$\frac{7/9}{10/9} = \frac{7}{10},$$

and the answer is $7 + 10 = 17$.

23. Let N be the number of arrangements of the six letters $ABCDEF$ such that if two letters are in the same row in the diagram below, they cannot be adjacent in the arrangement. For example, $EACFBD$ is a valid arrangement while $ADFBEC$ is not, because D and F are adjacent but both are in the same row in the diagram. Find the last two digits of N.

A

B C

D E F

Answer (20): We will first consider the row numbers of the letters in the arrangement; 1 representing A, 2 representing B and C, and 3 representing D, E, and F.

We will count the ways of ordering the string 122333 such that no 2's are adjacent and no 3's are adjacent. Let's do casework based on where the 3's are positioned.

Case 1: 3 _ 3 _ 3 _ or _ 3 _ 3 _ 3

In this case, the remaining slots for the 1's and 2's can be filled in any way because none of the slots to fill are adjacent. This gives us $2 \times 3 = 6$ ways.

Case 2: 3 _ 3 _ _ 3 or 3 _ _ 3 _ 3

Here, the only scenario that doesn't work is when the 2's are adjacent. Thus, there are $2 \times (3 - 1) = 4$ cases here.

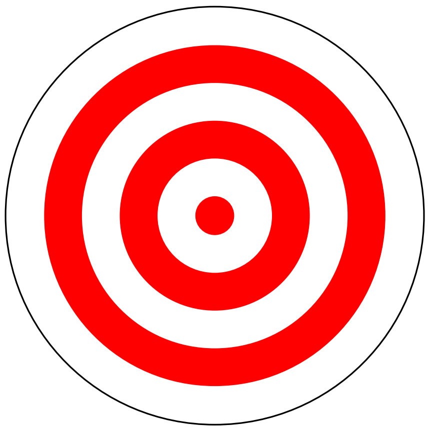
Combining the two cases above, we get 10 cases total.

Finally, we have to account for the order of the 2's and 3's. This leads to

$$N = 2! \cdot 3! \cdot 10 = 120.$$

The last two digits are 20.

24. A circular target at an archery range has a radius of 11 units, and consists of concentric circles of radii 1, 3, 5, 7, 9, and 11. The target's areas are painted red and white in an alternating fashion, starting with red in the middle circle. Bernadetta shoots two arrows at the target, and each arrow hits the target at a uniformly random position; the first arrow hits a red region, and the second arrow hits a white region. The probability that the first arrow is closer to the center of the target than the second arrow is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Answer (82): The total area is 121π. The total red area is

$$(1 + 25 - 9 + 81 - 49)\pi = 49\pi.$$

Thus, the white area is

$$121\pi - 49\pi = 72\pi.$$

We will do casework based on which red region the first arrow hit. There are three cases:

Case 1: First arrow lands in center-most red region. This occurs with probability $\frac{1}{49}$. Then, the probability that the second arrow lies farther from the center is 1 since all 3 white regions are outside of it. So, the probability for Case 1 is just $\frac{1}{49}$.

Case 2: First arrow lands in second red region from center. This occurs with probability

$$\frac{25-9}{49} = \frac{16}{49}.$$

Then, the probability that the second arrow lies farther from the center is

$$\frac{72-(9-1)}{72} = \frac{8}{9}.$$

So, the total probability for this case is

$$\frac{16}{49} \cdot \frac{8}{9} = \frac{128}{441}.$$

Case 3: First arrow lands in third red region from center. This occurs with probability $\frac{32}{49}$. Then, the probability that the second arrow lies farther from the center is

$$\frac{121-81}{72} = \frac{40}{72} = \frac{5}{9}.$$

So, the total probability for this case is

$$\frac{32}{49} \cdot \frac{5}{9} = \frac{160}{441}.$$

Thus, the total probability is

$$\frac{1}{49} + \frac{128}{441} + \frac{160}{441} = \frac{297}{441} = \frac{33}{49}.$$

The answer is $33 + 49 = 82$.

25. Find the remainder when

$$\left\lfloor\sqrt{1}\right\rfloor + \left\lfloor\sqrt{2}\right\rfloor + \left\lfloor\sqrt{3}\right\rfloor + \cdots + \left\lfloor\sqrt{2021}\right\rfloor + \left\lfloor\sqrt{2022}\right\rfloor$$

is divided by 100, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Answer (42): The terms equal to 1 have $1, 2, 3$ in the radical. The terms equal to 2 have $4, 5, \ldots, 8$ in the radical. In general, the terms equal to n have

$$n^2, n^2+1, \ldots, (n+1)^2-1$$

in the radical. Moreover, there are

$$(n+1)^2 - n^2 = 2n+1$$

such values.
Thus, the sum up to $\lfloor \sqrt{45^2-1} \rfloor = \lfloor \sqrt{2024} \rfloor$ is

$$\begin{aligned} & 1\cdot(2\cdot 1+1) + 2\cdot(2\cdot 2+1) + \cdots + 44\cdot(2\cdot 44+1) \\ = \; & 2\cdot(1^2+2^2+\cdots+44^2) + (1+2+\cdots+44) \\ = \; & 2\cdot\frac{44\cdot 45\cdot 89}{6} + \frac{44\cdot 45}{2} \\ = \; & 58740 + 990 \\ = \; & 59730. \end{aligned}$$

Now, from this sum we subtract off

$$\left\lfloor \sqrt{2023} \right\rfloor + \left\lfloor \sqrt{2024} \right\rfloor = 44 + 44 = 88,$$

and get

$$59730 - 88 = 59642.$$

The answer is 42.