Pi Math Contest Euler Division

2022 Solutions

The problems and solutions in this contest were proposed by:

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Solutions

1. What is $1 + 2 \times 3$?

   Answer (7): Following order of operations, this is

   $$1 + (2 \times 3) = 1 + 6 = 7.$$

2. How many multiples of 2 are between 1 and 9?

   Answer (4): The multiples of 2 (even numbers) between 1 and 9 are:

   $$2, 4, 6, 8.$$

   So, there are 4.

3. Anna is the 50th person in a line with 100 people. What is the positive difference between the number of people after her and the number of people before her?

   Answer (1): There are $100 - 50 = 50$ people after her and 49 people before her. The difference is $50 - 49 = 1$.

4. What is
$$8 \times \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right)?$$

Answer (7): Distributing the 8 into each term, we get
$$\frac{8}{2} + \frac{8}{4} + \frac{8}{8} = 4 + 2 + 1 = 7.$$

5. Emily was born on March 14, 2017. How many years old was she on January 1, 2022?

Answer (4): Her 4th birthday is March 14, 2021. Her 5th birthday is March 14, 2022. January 1, 2022 is between these two dates, so she was 4 on January 1, 2022.

6. The product of the digits of Alice's favorite two-digit number is 16. How many possibilities are there for Alice's favorite two-digit number?

Answer (3): There are 5 ways to write 16 as a product of two positive integers:
$$1 \times 16, \quad 2 \times 8, \quad 4 \times 4, \quad 8 \times 2, \quad 16 \times 1.$$
Among these, the first and the last contain 16 which is not a digit. The remaining three give one possibility each. So, there are 3 possibilities for Alice's favorite number: 28, 44, and 82.

7. A rectangle has a side of length 2 and a perimeter of 10. What is the area of this rectangle?

Answer (6): The perimeter has twice the length, which is 4, and twice the width. So, twice the width is $10 - 4 = 6$, and the width 3.

The area is then $2 \times 3 = 6$.

8. Ata exchanges 18 plastic bottles for 8 cents each. If he then exchanges as much of his money as possible for quarters, how many quarters will he have?

Answer (5): After exchanging his bottles, Ata will have $18 \times 8 = 144$ cents. This is more than $5 \times 25 = 125$ and less than $6 \times 25 = 150$, so he can get at most 5 quarters.

9. The problems on this test are numbered from 1 to 25. What percent of the problem numbers have two identical digits?

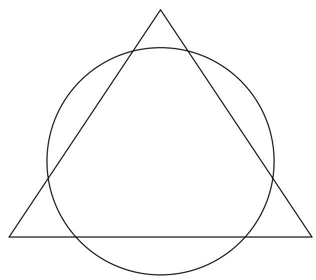
   Answer (8): The only possible numbers with identical digits are 11 and 22. Thus, there are 2 such problems out of 25 total, which is $\frac{2}{25} = \frac{8}{100}$, or 8%.

10. Compute the sum of the digits of 11^3.

   Answer (8):
   $$11^3 = 11^2 \times 11 = 121 \times 11 = 1331.$$
   The sum of its digits is
   $$1 + 3 + 3 + 1 = 8.$$

11. What is the greatest number of intersection points between a triangle and a circle?

   Answer (6): Any line segment can intersect a circle in at most 2 points, so the three line segments in a triangle can intersect the circle in at most 6 points. The following diagram shows an example with 6 intersection points, so the maximum is indeed 6.



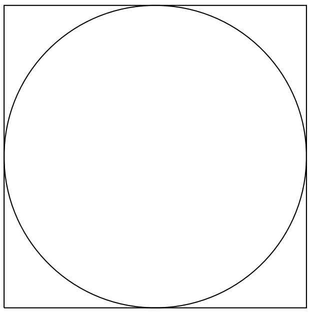
12. What is the result when the least common multiple of 4 and 6 is divided by the greatest common factor of 4 and 6?

   Answer (6): Multiples of 4 are $4, 8, 12, \ldots$ while multiples of 6 are $6, 12, 18, \ldots$. We see that the least common multiple (the smallest number appearing in both lists) is 12.

   On the other hand, the factors of 4 are 1, 2, and 4. The factors of 6 are 1, 2, 3, and 6. We find that the greatest common factor (largest number appearing in both lists) is 2.

Therefore, the answer is $\frac{12}{2} = 6$.

13. In the figure below, a circle is inscribed inside a square with side length 2. What is the area of the circle, to the nearest integer?



Answer (3): Since the side length of the square is 2, the radius of the circle is 1, so the area of the circle is $\pi \cdot 1^2 = \pi$. Approximating $\pi = 3.14$, the nearest integer is 3.

14. A room is in the shape of a rectangular prism. It has a base of 4.9 feet by 10.1 feet, and a volume of 201 cubic feet. What is the height of the room in feet, to the nearest integer?

Answer (4): Approximating the base dimensions as 5 feet and 10 feet and the volume as 200 cubic feet, the height is about $\frac{200}{5 \cdot 10} = 4$ feet.

15. Define the *awesomeness* of a number as the sum of all its positive integer divisors other than itself. For example, the awesomeness of 4 is $1 + 2 = 3$. Which positive digit has the largest awesomeness?

Answer (8): We compute the awesomeness of each number from 1 to 9. They are (in order): 0, 1, 1, 3, 1, 6, 1, 7, and 4. The largest of these is 7, the awesomeness of 8.

16. Triangle ABC is isosceles and angle B has a measure of 30 degrees. Candice computes the sum of the possible values of the degree measure of angle C. What is the sum of the digits of Candice's sum?

Answer (9): There are three cases as to which two sides are equal.

If $\angle A = \angle B$, then these two are $30°$ each, so the third angle is

$$\angle C = 180° - 30° - 30° = 120°.$$

If $\angle B = \angle C$, then $\angle C = 30°$.

If $\angle A = \angle C$, then their sum is $180° - 30° = 150°$. So, they are each $\frac{150°}{2} = 75°$.

Therefore, Candice's sum is

$$120 + 30 + 75 = 225.$$

The answer is $2 + 2 + 5 = 9$.

17. A triangle has area 8. A new triangle is formed by connecting the midpoints of the sides of the original triangle. What is the area of this new triangle?

    Answer (2): The lines divide the triangle into four smaller congruent triangles, each with half the side length of the original, so the answer is

$$\frac{8}{4} = 2.$$

18. The number $4096 = 2^{12}$ has 13 positive integer divisors: $1, 2, 2^2, 2^3, \ldots, 2^{12}$. How many positive integers are there whose square divides 4096?

    Answer (7): If a number's square divides 4096, the number itself must divide 4096. So, we have 13 candidates: $1, 2, 2^2, 2^3, \ldots, 2^{12}$. Among them, only 7 of them work, namely $1, 2, 2^2, 2^3, \ldots, 2^6$. Because, for the remaining numbers, their squares are larger than 2^{12}, and hence, don't divide 4096.

19. Helen flips a fair coin four times. How many possible sequences of flips do not have two consecutive tails?

    Answer (8): We will do casework based on the number of tails.

    If she flips zero tails, then she must have flipped the sequence HHHH.

    If she flips one tail, then there are 4 possible sequences: HHHT, HHTH, HTHH, and THHH.

    If she flips two tails, then there are 3 sequences in which they are not consecutive: THTH, THHT, and HTHT.

    If Helen flips at least 3 tails, then she must have two consecutive tails at some point.

    This makes a total of $1 + 4 + 3 = 8$ valid sequences.

20. Olivia thinks of a number. Her number is the number of divisors of 210 that are composite. What is the product of the digits of Olivia's number?

    Answer (1): $210 = 2 \cdot 3 \cdot 5 \cdot 7$, which has $2 \cdot 2 \cdot 2 \cdot 2 = 16$ positive divisors. All are composite except those that are prime or 1. Four are prime and 1 is neither prime nor composite, so $16 - 5 = 11$ divisors are composite, giving an answer of $1 \cdot 1 = 1$.

21. There are four people named Alice, Bob, Charlie, and Diane. For each person, there is a name tag with their name on it. How many ways can the four name tags be assigned to the four people such that each person gets one name tag and at least two people get their own?

    Answer (7): We will do casework based on the number of people who get their own tags.

    If exactly two people received their own name tag, then the other two must have swapped name tags. There are 6 ways for this to happen. Using the initials for the names, the two who get the correct name tags can be AB, AC, AD, BC, BD, or CD.

    It is impossible for exactly three people to receive their own name tag, as the only name tag left for the fourth person would be their own.

    There is 1 way for all people to get their own name tag.

    We conclude that the answer is $1 + 6 = 7$.

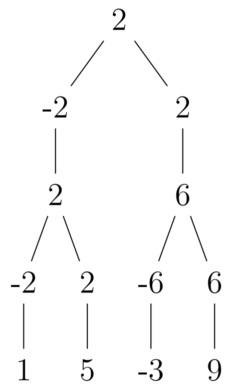
22. The letters of the word EULER are arranged randomly. What is the reciprocal of the probability that the vowels appear in alphabetical order *and* the consonants appear in alphabetical order?

    Answer (6): The vowels in EULER are E, E, and U. They are arranged in alphabetical order whenever the U appears last, which happens with probability $\frac{1}{3}$. The consonants in EULER are L and R. They are arranged in alphabetical order when the L appears first which happens with probability $\frac{1}{2}$. Thus the probability that both the vowels and consonants appear in alphabetical order is $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$, which gives an answer of 6.

23. Ayesha thinks of a positive number. She subtracts three and takes the absolute value of the result. Then, she subtracts four from that and takes the absolute value of the result. She ends up with 2. How many possible numbers could Ayesha have started with?

Note: The absolute value of a number is the distance between that number and 0. For example, the absolute value of -2 is 2, because the distance between -2 and 0 is 2.

Answer (3): We reverse the steps. For example, in the last step, Ayesha takes the absolute value and get 2, so the number before this step could be -2 or 2. Before that she subtracts 4, so we add 4 to find the prior number, and so on. We get the following:



So, she could have started with one of the following numbers: 1, 5, -3, and 9. Only one of them is negative and the remaining 3 are positive.

Alternate Solution: For a number x, let $|x|$ denote its absolute value.

Let a be the number that Ayesha starts with. Either

$$|x-3|-4=2,$$

in which case $x=9$ or $x=-3$, or

$$|x-3|-4=-2,$$

in which case $x=5$ or $x=1$.

Three of the four solutions are positive, so we have an answer of 3.

24. A *geometric sequence* is a sequence in which the ratio of successive terms is constant. For example, 1, 2, 4 is a geometric sequence. How many three-digit numbers with different digits have their digits form a geometric sequence when written in order?

Answer (8): Since the digits must be different, they can either be increasing (if the common ratio is larger than 1) or decreasing (if the common ratio is less than 1). We will first find the ones with increasing digits and reverse their digits to find the ones with decreasing digits.

Doing casework based on the first two digits, we end up with the following 4 numbers with increasing digits:

$$124, \quad 139, \quad 248, \ 469.$$

Reversing the digits, we get 4 more numbers with decreasing digits:

$$421, \quad 931, \quad 842, \ 964.$$

Hence, the answer is 8.

25. Rohan draws three distinct circles on a piece of paper such that no circle touches an edge of the paper. Let N be the number of regions into which they divide the paper. For example, if he draws three non-intersecting circles, there would be four regions; inside the first circle, inside the second circle, inside the third circle, and outside all circles. How many possible values are there for N?

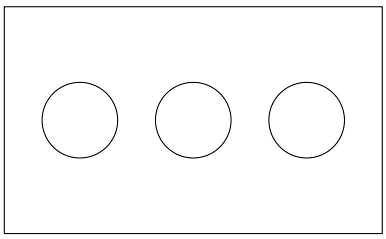
    Answer (5): Without any circles we start with 1 region, the paper itself. Each circle drawn adds at least one more region, so after three circles, we will have at least $1 + 3 = 4$ regions.

    On the other hand, by trying various possibilities, one can see that the largest number of regions is 8.

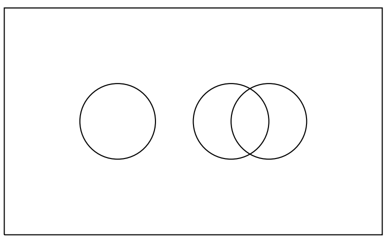
    Thus, the candidates for N are 4, 5, 6, 7, and 8.

    The diagrams below show that each of these 5 numbers are possible.

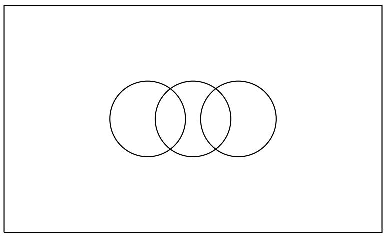
    4 regions:



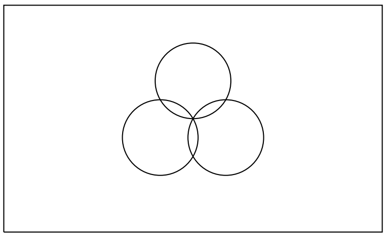
    5 regions:



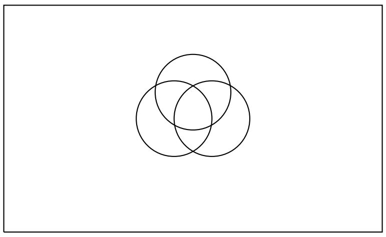
6 regions:



7 regions:



8 regions:



Hence, the answer is 5.